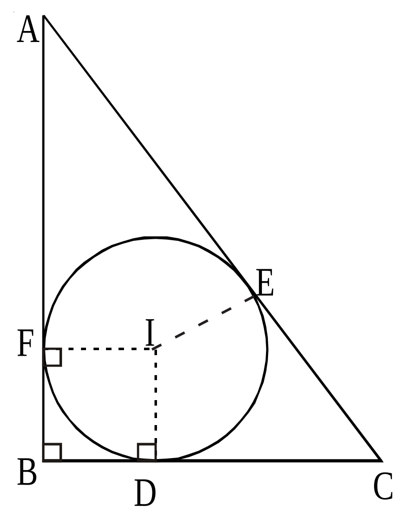
Mathematics Olympiad Class Room –

Problems of regional, national and international level of Mathematics Olympiad specific to certain mathematical concepts are discussed here. For this time, problems based on inradius have been taken up the discussion :

Inradius of a Triangle

Pankaj Agarwal



Property I : If ABC is a right-angled triangle, right-angled at B, the diameter of the circle inscribed in ΔABC equals AB+BC-AC.

Proof: Let I be the incentre of ΔABC and let the circle touch sides BC, CA and AB at D,E,F respectively.

Since IF = ID=r (radius of the circle) and

$\angle IFB=\angle FBD=\angle IDB=90^0$.

So, FIDB is a square of side 'r'.

If AB=c, BC=a and CA=b, then

AC=AE+EC

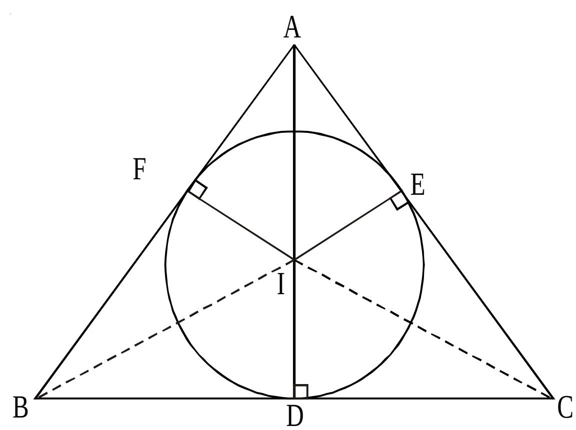
=AF+DC

=(AB-FB)+(BC-BD)

i.e., AC = AB+BC-2r

or 2r = AB+BC-AC

or 2r = c+a-b



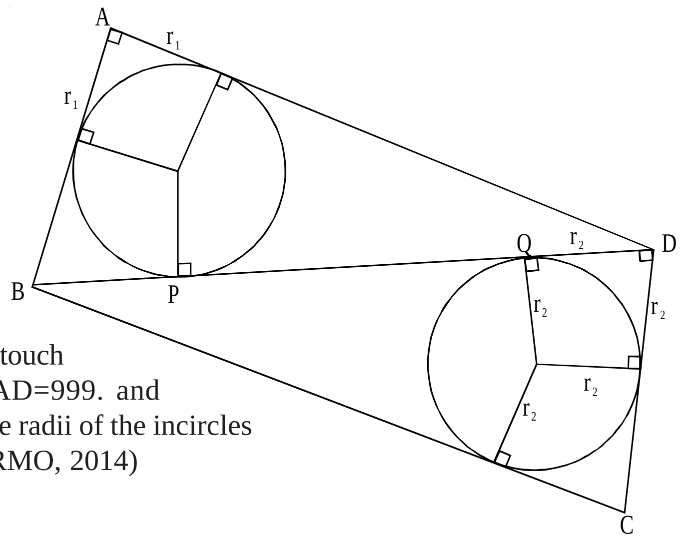
Property 2 : If Δ is the area of triangle ABC whose semi-perimeter is 's' and in radius 'r', then Δ= rs.

Proof: Let D,E,F be respectively the points at which the incircle touches the sides BC,CA and AB of triangle ABC. If I is the in-centre of the triangle ABC with area Δ, then

Δ= Area of triangle AIB+area of triangle BIC + area of triangle CIA

$= \frac{1}{2}.r.BA + \frac{1}{2}.r.BC + \frac{1}{2}.r.CA$

$= r.\left(\frac{AB+BC+CA}{2}\right) = rs$



Solved Examples

(1) Let ABCD be a convex quadrilateral with $\angle DAB = \angle BDC = 90^0$. Let the incircles of triangles ABD and BCD touch BD at P and Q respectively. If AD=999. and PQ=200, then what is the sum of the radii of the incircles of triangles ABD and BDC? (Pre-RMO, 2014)

Solution :

We know,

$2r_1 = AB + AD - BD$

$= (r_1 + BP) + 999 - (BP + PQ + QD)$

$\Rightarrow r_1 = 999 - 200 - r_2$

$\Rightarrow r_1 + r_2 = 799$

(2) Prove that the inradius of a right-angled triangle with integer sides is an integer. (RMO, 1999)

Solution : If 'r' is the in-radius of triangle ABC, right-angled at B and AB=c, BC=a, CA=b, then $2r = c + a - b$

Since a, b, c are integers, we just need to prove that c+a-b is even i.e., we need to prove that out of a, b, c the two cases are not possible

Case (i) : Exactly one of a, b, c, is odd.

This is obviously not possible as $b^2 = a^2 + c^2$. If exactly one of them is odd, the other two must be even.

But sum and difference of any two even numbers is even. So, this case is not possible.

Case (ii) : All of a, b, c are odd.

This is again obviously not possible as $b^2 = a^2 + c^2$ and then sum of two odd numbers is always even.

Hence proved.

(3) Determine the side lengths of a right triangle if they are integers and the product of the leg's lengths equals three times the perimeter. (Romania, 1999)

Solution : Let the sides be a, b, c with $a^2+c^2=b^2$ and in-radius of the triangle be 'r'. Also, let the area of the triangle be Δ.

Now, ac=3(a+b+c)............(1)

$$\Rightarrow \frac{1}{2}ac = 3.\left(\frac{a+b+c}{2}\right)$$

$\Rightarrow \Delta=3s$

$\Rightarrow rs=3s.$

$\Rightarrow r=3$............(2)

$\therefore$ 2r=a+c-b becomes

6=a+c-b.........(3)

$\Rightarrow (a+c)^2=(b+6)^2$

$\Rightarrow 2ac = 12b+36$

$\Rightarrow$ ac=18+6(a+c-6) (from eq^n (3))

$\Rightarrow$ ac-6(a+c)+36=18

$\Rightarrow$ (a-6) (c-6)=18 ....(4)

From eq^n (3), we know that a>6 and c>6 ($\because$ a-b<o and c-b<o)

So, eq^n (4) gives

a-6=1, c-6=18

or a-6=2, c-6=9

or a-6=3, c-6=6

or a-6=6, c-6=3

or a-6=9, c-6=2

or a-6=18, c-6=1

i.e., (a, b, c,) = (7, 25, 24), (8, 17, 15), (9, 15, 12), (12, 15, 9), (15, 17, 8), (24, 25, 7)

(4) The inradius of a triangle is 1 unit. The sides of the triangle as well as the semi perimeter of the triangle are all integers. Find the sides of the triangle.

Solution : Let the sides of the triangle be a, b, c with semi-perimeter $s=\frac{a+b+c}{2}$ and areaΔ.

Now, Δ=rs=s

$$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = s$$

$\Rightarrow$ (s-a) (s-b) (s-c) = s=(s-a) + (s-b)+(s-c)

Put s-a=x; s-b=y and s-c=z

So, xyz=x+y+z .....(1)

Without loss of generality we assome $x \geq y \geq z$.

So, eq^n (1) becomes

$xyz \leq x+x+x \Rightarrow yz \leq 3$

Case I : If yz=1, then y=z=1 and we get x=x+1+1 (from eq^n(1)) which is impossible.

Case II : If yz=2, then y=2, z=1 gives

2x = x+2+1 (from eq^n (1)) i.e., x=3

So, s-a=3, s-b=2, s-c=1 which gives s=(s-a)+(s-b)+(s-c)=6

Hence, a=3, b=4, c=5

Case II : If yz=3, then y=3, z=1 gives

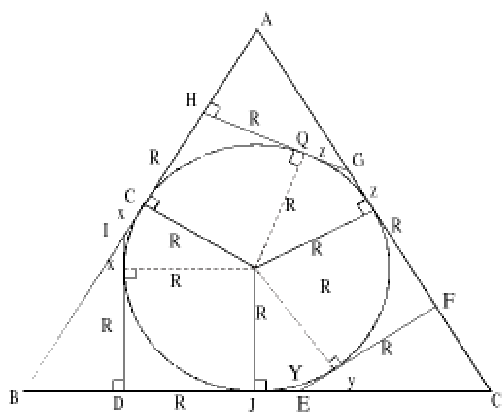
3x = x+3+1 (from eq^n(1) i.e., x=2<y

which contradicts $x \geq y \geq z$.

So, the sides can only be 3, 4, 5.

(5) A circle of radius R is incribed into an actute triangle,. Three tangents to the circle split the triangle into three right triangles and a hexagon that has perimeter Q. Find the sum of diameters of circles inscribed into the three right triangles. (Tournament of Towns, Juniors, A-Level, 2006)

Solution :



From the figure,

Q=R+x+x+R+R+y+y+R+R+z+z+R

i.e. Q-6R=2(x+y+z).....(1)

If r_1, r_2, r_3 are the in radii of ΔBDI, ΔECF and ΔAHG respectively, then

$2r_1$ = BD+DI-BI

= (BJ-R)+(R+x)-(BJ-x) ($\because$ BC=BJ)

= 2x

i.e., r_1=x

Similarly, r_2=y and r_3=z

$\therefore 2r_1+2r_2+2r_3 = 2x+2y+2z$

=Q-6R

Problems for Practice

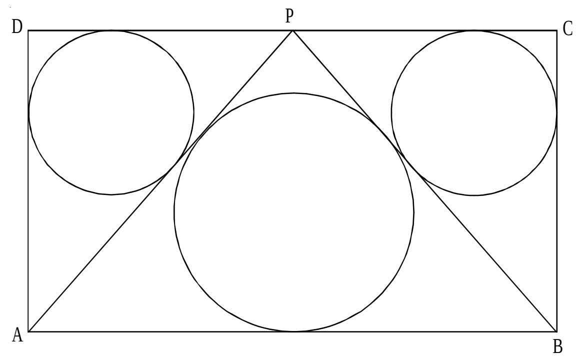
(1) In rectangle ABCD, AB=8 and BC=20. Let P be a point on AD such that $\angle BPC=90^0$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1+r_2+r_3$? (Pre-RMO, 2015)

(2) In the triangle ABC, the incircle touches the sides BC, CA and AB respectively at D,E and F. If the radius of the incircle is 4 units and if BD, CE and AF are consecutive integers, find the sides of the triangle ABC. (RMO, 1994)

(3) What is the minimal area of a right-angled triangle whose inradius is 1 unit? (RMO, 2014)

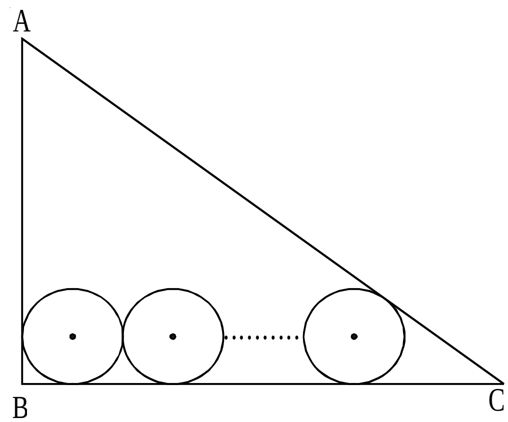
(Hint : If $x > 0, y > 0$, then $x + y \geq 2\sqrt{xy}$)

(4) In the figure, ABCD is a rectangle. Triangle PAB is isosceles. The radius of each of the smaller circles is 3cm and the radius of the bigger circle is 4 cm. Find the length and breadth of the rectangle. (Bhaskara Contest, Final Test, 2008)



(5) In the given figure, AB=6, BC=8, $\angle ABC=90^0$. There are 'K' congruent circles of radius 'r', each of which touches the side BC. Each circle excluding the first and last touches two circles. The first circle touches side AB while the last one touches side AC. If 'r' is an integer, then find all the possible values of K. Assume $K \geq 1$.

(Hint : Use the technique used in deriving property)



<u>Answer Key</u>

Q. No. (1) $r_1+r_2+r_3=8$

Q. No. (2) AB=14, BC=13, CA=15

Q. No. (3) Area $\geq 3+2\sqrt{2}$

Q. No. (4) Length = 24; breadth=9

Q. No. (5) K=1 or K=3

Mr. Pankaj Agarwal did his BE in Mechanical Engineering from Jorhat Engineering College, Assam. He started teaching mathematics in a coaching institute at Guwahati. Now, he is a senior faculty of Mathematics in a reputed coaching institute at Delhi.