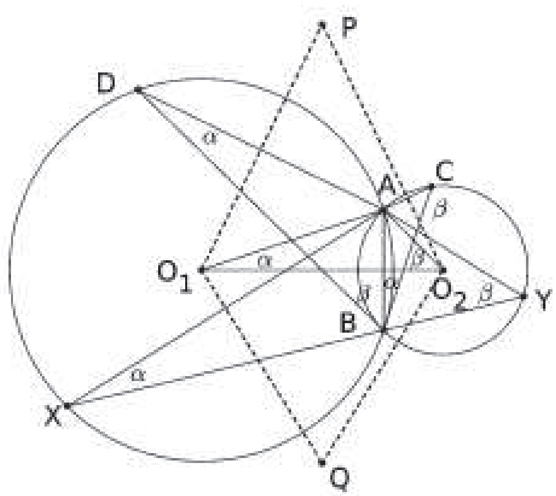
Problems and Solutions to INMO-2020

1. Let Γ_1 and Γ_2 be two circles of unequal radii, with centres O_1 and O_2 respectively, in the plane intersecting in two distinct points *A* and *B*. Assume that the centre of each of the circles Γ_1 and Γ_2 is outside the other. The tangent to Γ_1 at *B* intersects Γ_2 again in *C*, different from *B*; the tangent to Γ_2 at *B* intersects Γ_1 again in *D*, different from *B*. The bisectors of $\angle DAB$ and $\angle CAB$ meet Γ_1 and Γ_2 again in *X* and *Y*, respectively, different from *A*. Let *P* and *Q* be the circumcentres of triangles *ACD* and *XAY*, respectively. Prove that *PQ* is the perpendicular bisector of the line segment O_1O_2.

Solution:



Let $\angle CBA = \alpha$ and $\angle DBA = \beta$. Then $\angle BDA = \alpha$ and $\angle BCA = \beta$. We also observe that $\angle AO_1O_2 = (\angle AO_1B/2) = \alpha$ and, simiarly, $\angle AO_2O_1 = \beta$. Hence

$$\angle O_1AO_2 = 180^0 - (\alpha+\beta).$$

We also have

$$\angle PO_1A = \frac{\angle DO_1A}{2} = \frac{2\angle DBA}{2} = \angle DBA = \beta.$$

Hence $\angle PO_1O_2 = \angle PO_1A + \angle AO_1O_2 = \beta + \alpha$. Similarly, we can get $\angle PO_2O_1 = \alpha + \beta$. It follows that P lies on the perpendicular bisector of O_1O_2.

Now we observe that

$$\angle XQY = 360^0 - 2\angle XAY = 360^0 - 2(180^0 - \alpha - \beta) = 2(\alpha+\beta).$$

This gives

$$\angle O_1QO_2 = \frac{1}{2}(\angle XQA + \angle YQA) = \frac{\angle XQY}{2} = \alpha + \beta.$$

This shows that A, O_1, O_2, Q are concyclic. We also have

$$\angle ABX = \angle ABD + \angle DBX = \beta + \angle DAX = \beta + \frac{\angle DAB}{2};$$

$$\angle ABY = \angle ABC + \angle CBY = \alpha + \angle CAY = \alpha + \frac{\angle BAC}{2}.$$

Adding we obtain

$$\angle ABX + \angle ABY = \alpha + \beta + \frac{1}{2}(\angle DAB + \angle BAC) = \alpha + \beta + (180^0 - \alpha - \beta) = 180^0.$$

Hence X, B, Y are collinear. Now

$$\angle QAX = \frac{1}{2}(180^0 - \angle AQX) = 90^0 - \beta;$$

$$\angle XAO_1 = \frac{1}{2}(180^0 - \angle XO_1A) = 90^0 - \frac{1}{2}(360^0 - 2\angle ABX) = \angle ABX - 90^0.$$

Hence

$$\angle QAO_1 = 90^0 - \beta + \angle ABX - 90^0 = \angle ABX - \beta = \frac{\angle DAB}{2} = \frac{\angle O_1AO_2}{2}.$$

This shows that AQ bisects $\angle O_1AO_2$ and therefore the chords QO_1 and QO_2 subtend equal angles on the circumference of the circle passing through QO_2AO_1. Hence $QO_2 = QO_1$. This means Q lies on the perpendicular bisector of O_1O_2.

Combining, we get that PQ is the perpendicular bisector of O_1O_2.

2. Suppose $P(x)$ is a polynomial with real coefficients satsfying the condition $P(\cos\theta+\sin\theta)=P(\cos\theta-\sin\theta)$, for every real θ. Prove that $P(x)$ can be expressed in the form

$$P(x)=a_0+a_1(1-x^2)^2+a_2(1-x^2)^4+...+a_n(1-x^2)^{2n},$$

for some real numbers $a_0, a_1, a_2, ..., a_n$ and nonnegative integer n.

Solution: Changing θ to $\theta-\pi/2$, we see that

$$P(\sin\theta+\cos\theta)=P(\sin\theta-\cos\theta)$$

This shows that $P(x)=P(-x)$ for all $x\in\left[-\sqrt{2},\sqrt{2}\right]$ and as P is a polynomial, in fact,

$$P(x)=P(-x)$$

for all $x\in\mathbb{R}$. Hence $P(x)$ is an even polynomial; $P(x)=Q(x^2)$ for some polynomial $Q(x)$. This gives

$$Q(1+\sin(2\theta))=P(\cos\theta+\sin\theta)=P(\cos\theta-\sin\theta)=Q(1-\sin(2\theta)).$$

Taking $t=\sin(2\theta)$, we see that $Q(1+t)=Q(1-t)$. Hence $Q(0)=Q(2)$

Consider $Q(t)-Q(0)$. This vanishes both at $t=0$ and $t=2$. Hence $t(2-t)$ is a factor of $Q(t)-Q(0)$. We obtain

$$Q(t)-Q(0)=t(2-t)h(t)$$

for some polynomial $h(t)$. Using $Q(1+t)=Q(1-t)$, it follows that $h(1+t)=h(1-t)$. Hence by induction we get

$$Q(t)=\sum_{k=0}^{n} b_k t^k (2-t)^k .$$

Hence

$$P(x)=Q(x^2)=\sum_{k=0}^{n} b_k \left(x^2(2-x^2)\right)^k=\sum_{k=0}^{n} b_k \left(1-(1-x^2)^2\right)^k .$$

Using binomial theorem, we can write this as

$$P(x)=\sum_{k=0}^{n} a_k\left(1-x^2\right)^{2k},$$

for some coefficients $a_k, 0 \le k \le n$.

3. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $S \subseteq X$ be such that any nonnegative integer n can be written as $p + q$ where the nonnegative integers p, q have all their digits in S. Find the smallest possible number of elements in S.

Solution: We show that 5 numbers will suffice. Take $S = \{0, 1, 3, 4, 6\}$. Observe the following splitting:

n	a	b
0	0	0
1	0	1
2	1	1
3	0	3
4	1	3
5	1	4
6	3	3
7	3	4
8	4	4
9	3	6

Thus each digit in a given nonnegative integer is split according to the above and can be written as a sum of two numbers each having digits in S.

We show that $|S|>4$. Suppose $|S|\le 4$. We may take $|S|=4$ as adding extra numbers to S does not alter our argument. Let $S = \{a, b, c, d\}$. Since the last digit can be any one of the numbers 0, 1, 2, . . . , 9, we must be able to write this as a sum of digits from S, modulo 10. Thus the collection

$$A=\{x+y \pmod{10} \mid x, y\in S\}$$

must contain $\{0, 1, 2, . . . , 9\}$ as a subset. But A has at most 10 elements $\left(\binom{4}{2}+4\right)$. Thus each element of the form $x + y$ (mod 10), as x, y vary over S, must give different numbers from $\{0, 1, 2, . . . , 9\}$.

Consider $a + a$, $b + b$, $c + c$, $d + d$ modulo 10. They must give 4 even numbers. Hence the remaining even number must be from the remaining 6 elements obtained by adding two distinct members of S. We may assume that even number is $a + b$ (mod 10). Then a, b must have same parity. If any one of c, d has same parity as that of a, then its sum with a gives an even number, which is impossible. Hence c, d must have same parity, in which case $c + d$ (mod 10) is even, which leads to a contradiction. We conclude that $|S|\ge 5$.

4. Let $n\ge 3$ be an integer and let $1<a_1\le a_2\le a_2\le a_3\le ...\le a_n$ be n real numbers such that $a_1+a_2+a_3+...+a_n=2n$. Prove that

$$a_1a_2...a_{n-1}+a_1a_2...a_{n-2}+...+a_1a_2+a_1+2\le a_1a_2...a_n.$$

Solution: We use Chebyshev's inequality. Observe

$$n\left(a_1a_2...a_{n-1}+a_1a_2...a_{n-2}+...+a_1+1\right)$$

$$=\left(a_1a_2...a_{n-1}+a_1a_2...a_{n-2}+...+a_1+1\right)\left(\left(a_{n-1}-1\right)+\left(a_{n-1}-1\right)+...+\left(a_1-1\right)\right)$$

$$\leq n\left(a_1a_2...a_{n-1}\left(a_n-1\right)+...+a_1\left(a_2-1\right)+1\left(a_1-1\right)\right)$$

$$\leq n\left(a_1a_2...a_n-1\right).$$

It follows that

$$a_1a_2...a_{n-1}+a_1a_2...a_{n-2}+...+a_1+1\leq a_1a_2...a_n-1.$$

This gives the required inequality.

5. Infinitely many equidistant parallel lines are drawn in the plane. A positive integer $n\geq 3$ is called *frameable* if it is possible to draw a regular polygon with n sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.

(a) Show that 3, 4, 6 are *frameable*.

(b) Show that any integer $n\geq 7$ is not frameable.

(c) Determine whether 5 is *frameable*.

Solution: For n = 3, 4, 6 it is possible to draw regular polygons with vertices on the parallel lines (note that when we show a regular hexagon is a framed polygon, it includes the equilateral triangle case).

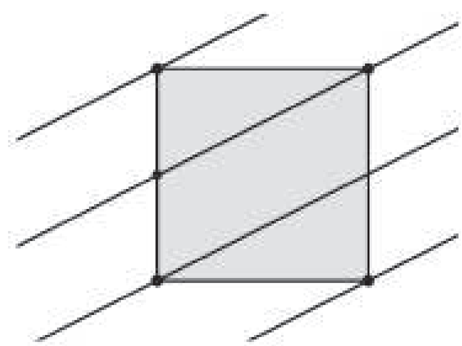


Figure 1:

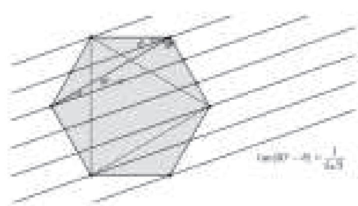


Figure 2:

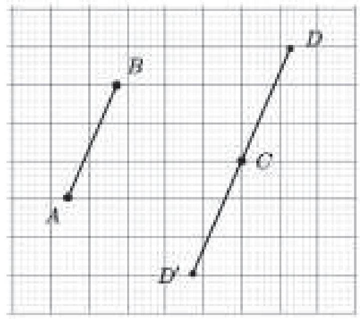


Figure 3:

We will prove that it is not possible for $n \geq 7$. In fact, we prove a stronger statement that we can not draw other polygons with vertices on the lines (even if we allow more than one vertex to lie on the same line).

First observe that if *A*, *B* are points on the lines and *C* is another point on a line, if we locate

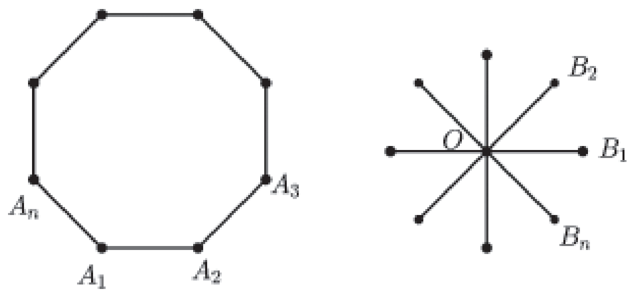


Figure 4:

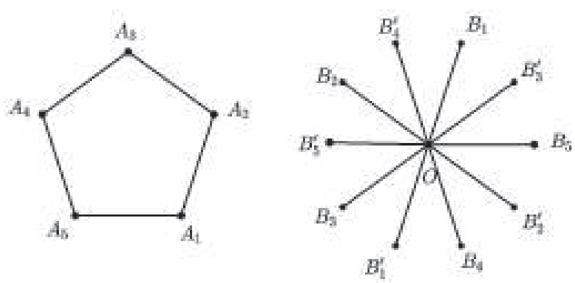


Figure 5:

point *D* such that *CD* is parallel and equal to *AB*, then *D* also lies on a line. Suppose that we have a regular polygon $A_1A_2...A_n$, where $n \geq 6$, with all the vertices on the grid lines. Choose a point O on a grid line and draw segments OB_i equal and parallel to A_iA_{i+1}, for $i = 1, 2, ..., n-1$ and OB_n parallel and equal to A_nA_1. The points B_i also lie on the grid lines and form a regular polygon with n sides. Consider the ratio $k = \frac{B_1B_2}{A_1A_2}$. Since $n > 6$, the $\angle B_1OB_2 < 360^0 / 6$ and hence is the smallest angle in the triangle B_1OB_2 (note that the triangle B_1OB_2 is isosceles). Thus $k < 1$. Hence starting with a polygon with vertices on grid lines, we obtain another polygon with ratio of side lengths $k < 1$. Repeating this process, we obtain a polygon with vertices on grid lines with ratio of sides k^m for any m. This is a contradiction since the length of the side of a polygon with vertices on grid lines can not be less than the distance between the parallel lines.

Thus for $n > 6$, we can not draw a polygon with vertices on the grid lines.

The above proof fails for $n = 5$. In this case, draw $OB_1, OB_1^{/}$ parallel and equal to A_1A_2, in opposite directions (see Figure 5), and similarly for other sides. Then we obtain a regular decagon with vertices on the grid lines and we have proved that this is impossible.

6. A *stromino* is a 3 × 1 rectangle. Show that a 5 × 5 board divided into twenty-five 1 × 1 squares cannot be covered by 16 *strominos* such that each *stromino* covers exactly three unit squares of the board and every unit square is covered by either one or two *strominos*. (A *stromino* can be placed either horizontally or vertically on the board.)

Solution: Suppose on the contrary that it is possible to cover the board with 16 strominos such that each unit square is covered by either one or two strominos. If there are k squares that are covered by exactly one stromino then $2(25 - k) + k = 163 = 48$ and hence $k = 2$. Thus there are exactly two squares which are covered by only one stromino. We colour the board with three colours red, blue, green as follows. The square corresponding to the i-th row and the j-th column is coloured red if $i + j = 0 \pmod 3$, green if $i + j = 1 \pmod 3$ and blue otherwise. Then there are 9 red squares, 8 green squares and 8 blue squares. Note that each stromino covers exactly one square of each colour. Therefore the two squares that are covered by only one stromino are both red. For each such square $i + j = 0 \pmod 3$ where i and j are its row and column number.

We now colour the board with a different scheme. We colour the square corresponding to the i-th row and the j-th column red if $i - j = \pmod 3$, green if $i - j = 1 \pmod 3$ and blue otherwise.

Again, there are 9 red squares and hence the two squares covered by only one stromino are both red. For each such square $i - j = 0 \pmod 3$ where i and j are its row and columne number Thus, each of the two squares covered by only one stromino satisfies $i + j = 0 \pmod 3$ and $i - j = 0 \pmod 3$ where i and j are its row and column number. This implies that $i = j = 3$. This is a contradiction because there is only one such square.