Methematical Talent Search Corner

Problems and Solutions of RMO-2019

1. Suppose x is a nonzero real number such that both x^5 and $20x + \dfrac{19}{x}$ are rational numbers. Prove that x is a rational number.

   Solution : Since x^5 is rational, we see that $(20x)^5$ and $(x/19)^5$ are rational numbers. But

   $$(20x)^5 - \left(\frac{19}{x}\right)^5 = \left(20x - \frac{19}{x}\right)\left((20)^4 + \left(20^3.19\right)x^2 + 20^2.19^2 + \left(20.19^3\right)\frac{1}{x^2} + \frac{19^4}{x^4}\right).$$

   Consider

   $$\begin{aligned} T &= \left((20x)^4 + (20^3 \cdot 19)x^2 + 20^2 \cdot 19^2 + (20 \cdot 19^3)\frac{1}{x^2} + \frac{19^4}{x^4}\right) \\ &= \left((20x)^4 + \frac{19^4}{x^4}\right) + 20 \cdot 19\left((20x)^2 + \frac{19^2}{x^2}\right) + (20^2 \cdot 19^2). \end{aligned}$$

   Using $20x + (19)/^x$ is rational, we get

   $$(20x)^2 + \frac{19^2}{x^2} = \left(20x + \frac{19}{x}\right)^2 - 2 \cdot 20 \cdot 19$$

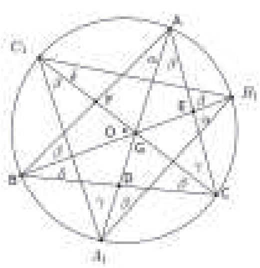
   is rational. This leads to

   $$(20x)^4 + \frac{19^4}{x^4} = \left((20x)^2 + \frac{19^2}{x^2}\right)^2 - 2 \cdot 20^2 \cdot 19^2$$

   is also rational. Thus T is a rational number and $T \neq 0$. We conclude that $20x$-$(19/x)$ is a rational number. This combined with the given condition that $20x + (19/x)$ is rational shows $2 \cdot 20x$ is rational. Therefore x is rational.

2. Let ABC be a triangle with circumcircle Ω and let G be the centroid of triangle ABC. Extend AG, BG and CG to meet the circle Ω again in A_1, B_1 and C_1, respectively. Suppose $\angle BAC = \angle A_1B_1C_1$, $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.

   Solution :

   Let $\angle BAA_1 = \alpha$ and $\angle A_1AC = ß$. Then $\angle BB_1A_1 = \alpha$. Using that angles at A and B_1 are same, we get $\angle BB_1C_1 = ß$. Then $\angle C_1CB = ß$. If $\angle ACC_1 = \gamma$, we see that $\angle C_1A_1A = \gamma$.

Therefore $\angle AA_1B_1 = ß$. Similarly, we see that $\angle B_1BA = \angle A_1C_1C = ß$ and $\angle B_1BC = \angle B_1C_1C = \delta$.

Since $\angle FBG = \angle BCG = ß$, it follows that FB is tangent to the circumcircle of ΔBGC at B. Therefore $FB^2 = FG \cdot FC$. Since $FA = FB$, we get $FA^2 = FG \cdot FC$. This implies that FA is tangent to the circumcircle of of ΔAGC at A. Therefore $\alpha = \angle GAF = \angle GCA = \gamma$. A similar analysis gives $\alpha = \delta$.

It follows that all the angles of ΔABC are equal and all the angles of $\Delta A_1B_1C_1$ are equal. Hence ABC and $A_1B_1C_1$ are equilateral triangles.

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3} \le \frac{1}{5abc}.$$

Solution : Observe that

$$a^2+b^3+c^3 = a^2(a+b+c)+b^3+c^3 = (a^3+b^3+c^3)+a^2(b+c) \ge 3abc+a^2b+a^2c.$$

Hence

$$\frac{a}{a^2+b^3+c^3} \le \frac{1}{3bc+ab+ac}.$$

Using AM-HM inequality, we also have

$$\frac{3}{bc}+\frac{1}{ca}+\frac{1}{ab} \ge \frac{25}{3bc+ca+ab}.$$

Thus we get

$$\frac{a}{a^2+b^3+c^3} \le \frac{1}{3bc+ab+ac} \le \frac{1}{25}\left(\frac{3}{bc}+\frac{1}{ca}+\frac{1}{ab}\right).$$

Similarly, we get

$$\frac{b}{b^2+c^3+a^3} \le \frac{1}{25}\left(\frac{3}{ca}+\frac{1}{ab}+\frac{1}{bc}\right)$$

and

$$\frac{c}{c^2+a^3+b^3} \le \frac{1}{25}\left(\frac{3}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

Adding, we get

$$\begin{aligned}\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3} &\le \frac{5}{25}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\\ &= \frac{1}{5abc}.\end{aligned}$$

4. Consider the following 3×2 array formed by using the numbers 1, 2, 3, 4, 5, 6:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers 7, 8, 9, . . . , $3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \quad \text{and} \quad \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2.$$

Solution : Consider the following extension:

$$\begin{pmatrix} 1 & 6 & 3+6 & 4+6 & 2+(2\cdot 6) & 5+(2\cdot 6) \\ 2 & 5 & 1+6 & 6+6 & 3+(2\cdot 6) & 4+(2\cdot 6) \\ 3 & 4 & 2+6 & 5+6 & 1+(2\cdot 6) & 6+(2\cdot 6) \end{pmatrix}$$

of

$$\begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$$

This reduces to

$$\begin{pmatrix} 1 & 6 & 9 & 10 & 14 & 17 \\ 2 & 5 & 7 & 12 & 15 & 16 \\ 3 & 4 & 8 & 11 & 13 & 18 \end{pmatrix}$$

Observe

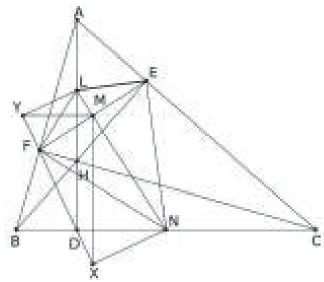
$$\begin{aligned} &1+6+9+10+14+17=57; \quad 1^2+6^2+9^2+10^2+14^2+17^2=703; \\ &2+5+7+12+15+16=57; \quad 2^2+5^2+7^2+12^2+15^2+16^2=703; \\ &3+4+8+11+13+18=57; \quad 3^2+4^2+8^2+11^2+13^2+18^2=703. \end{aligned}$$

Thus, in the new array, all row sums are equal and the sum of the squares of entries in each row are the same. Here $k = 6$ and we have added numbers from 7 to 18.

5. In a triangle ABC, let H be the orthocenter, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be midpoints of segments AH, EF, BC, respectively. Let X, Y be feet of altitudes from L, N on to the line DF. Prove that XM is perpendicular to MY.

Solution : Observe that AFH and HEA are right-angled triangles and L is the mid-point of AH. Hence $LF = LA = LE$. Similarly, considering the right triangles BFC and BEC, we get $NF = NE$. Since M is the mid-point of FE it follows that $\angle LMF = \angle NMF = 90^0$ and L, M, N are collinear. Since LY and NX are perpendiculars to XY, we conclude that $YFML$ and $FXNM$ are cyclic quadrilaterals. Thus

$$\angle FLM = \angle FYM, \text{ and } \angle FXM = \angle FNM.$$



We also observe that CFB is a right triangle and N is the mid-point of BC. Hence $NF = NC$. We get

$$\angle NFC = \angle NCF = 90^0 - \angle B.$$

Similarly, $LF = LA$ gives

$$\angle LFA = \angle LAF = 90^0 - \angle B.$$

We obtain

$$\angle LFN = \angle LFC + \angle NFC = \angle LFC + 90 - \angle B = \angle LFC + \angle LFA = \angle AFC = 90^0.$$

In triangles YMX and LFN, we have

$$\angle XYM = \angle FYM = \angle FLM = \angle FLN,$$

and

$$\angle YXM = \angle FXM = \angle FNM = \angle FNL.$$

It follows that $\angle YMX = \angle LFN = 90^0$?. Therefore $YM \perp MX$.

6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $gcd(a, b) = gcd(b, c) = gcd(c, d) = gcd(d, a) = 1$.

Solution : Let the given integers be $a_1, a_2, \ldots, a_{91}$. Take a 91 × 91 grid and color the cell at (i, j) black if $\gcd(a_i, a_j) = 1$. Then at least $2 \times 456 = 912$ cells are colored black. If d_i is the number of black cells in the ith column, then $\sum d_i \geq 912$. Now,

$$\begin{aligned}\sum_{i=1}^{91}\binom{d_i}{2} &\geq \frac{1}{2}\left[\frac{1}{91}\left(\sum_{i=1}^{91} d_i\right)^2 - \sum_{i=1}^{91} d_i\right]\\ &= \frac{1}{2\times 91}\left(\sum_{i=1}^{91} d_i\right)\left(\sum_{i=1}^{91} d_i - 91\right)\\ &\geq \frac{1}{2\times 91}\times 2\times 456\times(2\times 456-91)\\ &> \binom{91}{2}\end{aligned}$$

Since there are only $\binom{91}{2}$ distinct pairs of columns, there must be at least one pair of rows (u, v) that occur with two distinct columns s, t. Thus (u, s),(u, t),(v, s) and (v, t) are all black. Thus if the integers corresponding to the columns u, v, s, t are a, c, b, d respectively, then $gcd(a, b) = gcd(b, c) = gcd(c, d) = gcd(d, a) = 1$.

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