Methematical Talent Search Corner

Problem & Solution Mathematics Olympiad : 2019 (AAM)

Category-I : (For Classes- V & VI)

1. abcde is a five digit number. Two six digit numbers are formed by putting 9 to the left and right of it respectively. If the former is equal to 4 times of the latter, find a+b+c+d+e. 7

Soln.

By given condition,

9 abcde = 4 × abcde 9

i.e.

$$\begin{array}{r} a\ b\ c\ d\ e\ 9 \\ \times\ 4 \\ \hline 9\ a\ b\ c\ d\ e \end{array}$$

Comparing with digits in the bottom row–

e = 6, d = 7, c = 0, b = 3, a = 2

Thus abcde is 23076

Therefore a+b+c+d+e = 2+3+0+7+6

= 18

2. If you write all the first 100 natural numbers side by side in its natural order a large number will be formed. Now find the number of digits in the number so formed. Also find the sum of all the digits in the number. 7

Soln.

No. of digits from 1 through 9 = 9

No. of digits from 10 through 99 = (99-9)×2 = 180

No. of digits in 100 = 3

Total no. of digits in

numbers from 1 through 100 = 9+180+3

= 192

Next,

The no. of 0’ is 11

The no. of 1’s is 21

The no. of 2’s is 20

The no. of 3’s is 20

The no. of 4’s is 20

The no. of 5's is 20

The no. of 6's is 20

The no. of 7's is 20

The no. of 8's is 20

The no. of 9's is 20

Hence sum of all digits of the number composed of serial arrangement of numbers from 1 through 100

is $11\times+21\times1+20\times(2+3+4+5+6+7+8+9)$

$= 21+ 20\times44$

$= 21+ 880$

$= 901$

3. If the sum of seven consecutive natural numbers is 126, find the numbers. 6

Soln.

$126 \div 7 = 18$

So, the seven numbers might be around 19 let us try with 15, 16, 17, 18, 19, 20, 21

The sum of these numbers = 126

Thus, the consecutive numbers are

15, 16, 17, 18, 19, 20 and 21

4. Assign appropriate digits to the letters involved in the following two additions so that both of them remain correct in their digital values also. 7

$$\begin{array}{r} ONE \\ +\,ONE \\ \hline TWO \end{array} \qquad \begin{array}{r} ONE \\ +\,FOUR \\ \hline FIVE \end{array}$$

E+R=E => R=0

E+E=0 => 0 is even, but 0+0 is less then 10

Hence 0 is 4 or 2

If 0=2, E=1 or e=6

But it can been that E=1 is not possible

Therefore E=6

After a series of trial and error we see that N=3 or N=8

$\therefore$ One = 236 or One= 286

When One = 236, we obtain the sums as

$$\begin{array}{r} 236 \\ 236 \\ \hline 472 \end{array} \qquad \begin{array}{r} 236 \\ F2U0 \\ \hline 9516 \end{array} \quad \Rightarrow U = 8, F = 9$$

Thus $\begin{array}{r} 236 \\ 236 \\ \hline 472 \end{array}$, $\begin{array}{r} 236 \\ 9280 \\ \hline 9516 \end{array}$ give one solution.

Again ONE = 286 gives in

$$\begin{array}{r} 286 \\ 286 \\ \hline 572 \end{array}, \qquad \begin{array}{r} 286 \\ 92U0 \\ \hline 3496 \end{array} \quad \text{is } U = 1, F = 3$$

The other solution is

$$\begin{array}{r} 286 \\ 286 \\ \hline 572 \end{array}, \qquad \begin{array}{r} 286 \\ 3210 \\ \hline 3496 \end{array}$$

5. In the following addition sum, each of the ten digits is different and also the digits to be added in each column from top to bottom are in increasing order. Determine the digits in the sum 7

$$\begin{array}{r} A \\ +\quad B\,4 \\ +\quad C\,D\,E \\ \hline F\,G\,H\,I \end{array}$$

Soln.

A closer observation leads us to conclude that C=9 and 1 will be carried over to C=9 from B+D to get FG=10

Thus the sum becomes

$$\begin{array}{r} A \\ B\,4 \\ 9\,D\,E \\ \hline 1\,0\,H\,I \end{array}$$

Again A < 4 and E > 4, A, 4, E being in increasing order

After few trial and error steps we arrive at

A = 3 E = 5

B = 7, D = 8, H = 6, I = 2

i.e. sum is

$$\begin{array}{r} 3 \\ 7\ 4 \\ 9\ 8\ 5 \\ \hline 1\ 0\ 6\ 2 \end{array}$$

6. In a secret message if 'CDUQNWVG' is to be decoded as 'ABSOLUTE' how would you code the message ' NO WAY' 6

Soln.

Acording to the code

$A \rightarrow C$
$B \rightarrow D$
$S \rightarrow U$
$L \rightarrow N$
$U \rightarrow W$
$T \rightarrow V$
$E \rightarrow G$

Therefore

$N \rightarrow P$
$O \rightarrow Q$
$W \rightarrow Y$
$A \rightarrow C$
$Y \rightarrow A$

Hence, NO WAY is to be coded as PQ YCA

7. I am a three digit square number. If you divide me and the sum of my three digits by 3 and 5 you will find the remainder 1 in each case. Who am I? 6

Soln.

Three digit square numbers are–

100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625 676, 729, 784, 841, 900, 961

By condition, my units digit must be 1 more than 0 or 5 since I am divisible by 5. Therefore my units place will be 1 or 6. Hence, I am one of–

121, 196, 256, 361, 441, 576, 676, 841 and 961

Now

1+9+6 = 16 satisfies the other condition also.

9+6+1 = 16 also satisfies the other condition

Hence, I am 196 or 961

8. Follow the pattern given below and supply at least five terms to continue the pattern futher. 6

4, 6, 8, 9, 10, 12, 14, 15, 16, _, _, _, _.

Soln.

The terms in the sequence are all composite numbers. In other words the prime

numbers are not to be considered.
So, next five terms after 16 are–
18, 20, 21, 22, 24.

9. Supply the missing figures in the following multiplication

7

$$\begin{array}{r} * \; 5 \; 2 \; * \\ 3 \; * \\ \hline 7 \; * \; * \; 2 \;\; \\ * \; * \; * \; 8 \\ \hline 8 \; * \; * \; * \; * \end{array}$$

Soln.

For conveience *'s are replaced by letters like

$$\begin{array}{r} a \; 5 \; 2 \; b \\ 3 \; c \\ \hline 7 \; d \; e \; 2 \;\; \\ f \; g \; h \; 8 \\ \hline 8 \; j \; k \; l \; m \end{array}$$

From the multiplication, we can see
3×* leaves 2 in the unit place
Hence * must be 4
Sum becomes–

$$\begin{array}{r} a \; 5 \; 2 \; 4 \\ 3 \; c \\ \hline 7 \; 5 \; 7 \; 2 \;\; \\ f \; g \; h \; 8 \\ \hline 8 \; j \; k \; l \; m \end{array}$$

Now, $c \times 4 = 8 \quad \therefore c = 2$
Product becomes

$$\begin{array}{r} 2 \; 5 \; 2 \; 4 \\ 3 \; 2 \\ \hline 7 \; 5 \; 7 \; 2 \;\; \\ 5 \; 0 \; 4 \; 8 \\ \hline 8 \; 0 \; 7 \; 6 \; 8 \end{array}$$

10. Find the greatest number of four digits and the least number of five digits which when divided by 789 leave a remainder 5 in each case.

Soln.

The greatest four digit number is 9999.

Now 9999 divided by 789 leaves quotlent 12 and 531 as remainder. Thus, the greatest four digit number divisible by 789 is 9999-531 = 9468. Required greatest four digit number that leaves remainder 5 upon divided by 789 is 9468+5 = 9473

$$\begin{array}{r|l} & \quad 12 \\ \hline 789 & 10000 \\ & \underline{\ 789\ \ } \\ & \ \ 2110 \\ & \ \ \underline{1578} \\ & \ \ \ 532 \end{array}$$

Again the least five digit number 10000 divided by 789 leaves remainder 532. But 789-532 = 257

Therefore, the least five digit number divisible by 789 is 10000 + 257 = 10257. Hence, the least five digit number is leaving remainder 5 when divided by 789 is 10257 + 5 = 10262

Required numbers are 9473 and 10262

11. A sum of Rs. 22000 was distributed amount 60 students such that each senior student gets Rs. 500/- while each junior student gets Rs. 300/-. Find the numbers of senior and junior students among them. 6

Soln.

The amount required to distribute among 60 students at the rate of Rs. 300 per student is Rs. 300×60 = Rs. 1800

So, Rs. 22000 – Rs. 18000 = Rs. 4000 can be distributed to senior students at the rate of Rs. 200 per student.

Thus the number of senior student is 4000 ÷ 200 = 20

Hence the number of junior students is 40

Senior students 20, Junior students 40

12. A salesman bought a certain number of eggs for Rs. 186/- and sold some of them for Rs. 66/- without any profit. Show that he was still left with at least 20 eggs.7

Soln.

We have 186-66 = 120

and
$$\begin{array}{r|l} 2 & 66,\ 120 \\ \hline 3 & 33,\ \ 60 \\ \hline & 11,\ \ 20 \end{array}$$

The salesman can sell at a maximum rate Rs. 6 per egg.

Hence the minimum number of eggs left with the salesman is 120 ÷ 6 = 20

i.e. The salesman has at least 20 eggs for selling.

13. A vessel contains a mixture of 30 litres of water and milk in the ratio 7:3. How much milk must be added to the mixture so that the ratio of water and milk becomes 3:7? 7

Soln.

Water in the vessel is $\frac{7}{10} \times 30 = 21$ liters

Milk in the vessel is $\frac{3}{10} \times 30 = 9$ liters

After adding milk to the mixture,

Water : Milk = 3:7

i.e. 21 : Milk = 3:7

or Milk : 21 = 7:3

$\therefore$ Milk = $\frac{7}{3} \times 21 = 49$ liters

Amount of milk to be added is 49–9 = 40 litres.

14. Find the least square number which is divisible by 10, 16 and 24. 7

Soln.

$$\begin{array}{r|l} 2 & 10,\ 16,\ 24 \\ \hline 2 & 5,\ \ 8,\ 12 \\ \hline 2 & 5,\ \ 4,\ \ 6 \\ \hline & 5,\ \ 2,\ \ 3 \end{array}$$

$\therefore$ L C M of 10, 16 and 24

is $2 \times 2 \times 2 \times 5 \times 2 \times 3$

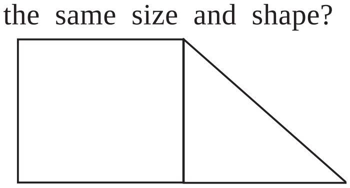
$= 2^2 \times 2^2 \times 5 \times 3$

Hence the least square number divisible by 10, 16

and 24 is $= 2^2 \times 2^2 \times 5^2 \times 3^2 \quad = 16 \times 425$

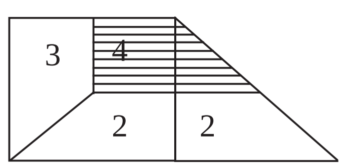
$= 6800$

15. The shape shown below is that of a square attached to half of another square of equal size divided diagonally. Can you divide it into four pieces all of precisely the same size and shape? 7



Soln.

Four pieces of same shape and size can be done as follows–



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Methematical Talent Search Corner

Category-II : (For Classes- VII & VIII)

1. Evaluate : $1^2-2^2+3^2-4^2+5^2-6^2+$ ...... $+ 99^2-100^2$ 6

Soln.

$1^2-2^2+3^2-4^2+5^2-6^2+ \ldots\ldots + 99^2-100^2$

$= (1+2)\ (1-2) + (3+4)\ (3-4) + \ldots + (99+100)\ (99-100)$

$= -1-2-3-4-\ldots -99-100$

$= -\ (1+2+3+\ldots. + 100)$

$= -\ 50 \times 101$

$= -\ 5050$

2. Simplify — 6

$$\frac{(1.2)^2 \times (0.05)^2 \div (0.25)^2}{(0.1)^2 \div (0.01)^2} \div 0.00288$$

Soln.

$$\frac{(1.2)^2 \times (0.05)^2 \div (0.25)^2}{(0.1)^2 \div (0.01)^2} \div 0.00288$$

$$= \frac{\left(\frac{1.2 \times 0.05}{0.25}\right)^2}{\left(\frac{0.1}{0.01}\right)^2} \div 0.00288$$

$$= \frac{\left(\frac{1.2 \times 0.01}{0.05}\right)^2}{10^2} \div 0.00288$$

$$= \frac{\left(\frac{1.2}{5}\right)^2}{100} \times \frac{1}{.00288}$$

$$= \frac{(.24)^2}{100} \times \frac{100000}{288}$$

$$= \frac{.24 \times .24}{100} \times \frac{100000}{288}$$

$$= \frac{\overset{2}{\cancel{24}} \times \cancel{24}}{100 \times 100} \times \frac{1000}{\underset{\cancel{12}}{\cancel{288}}}$$

$$= \frac{2}{10}$$

$= 0.2$

3. In the following multiplication sum, each of the digits from 1 through 9 appears exactly once in the multiplicand, multiplier and the product. One digit being known, supply the remaining digits. 7

$$\begin{array}{r} 2\ a\ b \\ \times \quad c\ d \\ \hline e\ f\ g\ h \end{array}$$

Soln.

Through trial and error, we can get the solutionas

$$\begin{array}{r} 2\ 9\ 7 \\ \times \quad 1\ 8 \\ \hline 5\ 3\ 4\ 6 \end{array}$$

4. How many 3 digit numbers are there for each of which 7 appears just once only. 6+1

Soln

The unit's place can be filled up in 10 different ways since any one of 0, 1, 2, 3, ..., 9 can be put in the units place. Whatever digit is put in unit's place the ten's place can again be filled up in 10 ways. So the unit's and ten's places can be filled in $10 \times 10 = 100$ ways.

Now, the hundred's place can be filled in 9 ways since 0 is not allowed in hundred's place.

Thus the total number of 3 digit numbers is $9 \times 10 \times 10 = 900$.

Similarly total no. of numbers with 7 occurring in none is $9 \times 9 \times 8 = 648$

Therefore, no. of three digit numbers with 7 occuring once, twice or thrice in each = 900–648=252

Let us count the numbers in which 7 is repeated twice or thrice.

There are 8 three digit numbers with 7 in both unit and ten's places.

There are 9 three digit numbers with 7 in unit's and hundred's places. Also there are 9 three digit numbers 7 in tens and hundred places.

Altogether there are 8+9+9 = 26 three digit numbers with 7 repeated twice.

Finally, there is one three digit number with 7 repeated thrice.

Therefore, number of three digit numbers with 7 occuring just once is 252-26-1 = 225

No of three digit numbers with 7 appearing only in units place is $9\times8=72$

Otherwise,

No of three digit numbers with 7 appearing only in ten's place is $9\times8=72$

No of three digit numbers with 7 appearing onlny is hundred place is $9\times9=81$

Therefore no. of three digit numbers with 7 appearing just once in each is 72+72+81=225.

5. What are the two natural numbers whose difference is 66 and the least common multiple is 360. 6+1

Soln.

The HCF of two numbers will be same as the HCF of the difference and LCM of the numbers.

Now difference of the number is 66

Their LCM is 360

$66 = 6 \times 11$

$360 = 6\times 60$

Hence, HCF of 66 and 360 in 6.

This means the HCF of the two numbers is also 6.

If a and b be the natural numbers then

$ab = HCF \times LCM = 6 \times 360 = 2160$

Now, $a-b = 66$

But $(a+b)^2 = (a-b)^2 + 4ab$

$= 66^2 + 4 \times 2160$

$= 12996$

$= 114^2$

$\therefore$ a+b = 114.

$$\therefore a=\frac{1}{2}\left[(a+b)+(a-b)\right]$$

$$=\frac{1}{2}[114+66]=\frac{1}{2}\times 180=90$$

$$\therefore b=90-66=24$$

The numbers are 90 and 24

6. Find two unequal numbers A and B such that A+n is a factor of B+n for all values of n from 1 to 11. 7

Soln.

Consider A=1. B=1×2×3× ..... ×11×12+1

For n=1, A+n=2, B+n=1×2×.....×12+1+1=2× (3×4×...×12+1)

∴ A+n | B+n for n=1

For n=2, A+2=3, B+2=3 (1×2+4×... ×12+1)

∴ A+n | B+n for n=2

n=11, A+11=12, B+11=12×(1×2×...×11+1)

∴ A+11 | B+11 for n=11

Thus the values of A and B are

1 and 1×2×2×3...×11×12+1 respectively.

7. Find the greatest prime number that will divide 12260 leaving remainder 17. 6

Soln.

12260-17 = 12243

Now

$$\begin{array}{r|l} 3 & 1\;2\;2\;4\;3 \\ \hline 7 & 4\;0\;8\;1 \\ \hline 11 & 5\;8\;3 \\ \hline & \;\;5\;3 \end{array}$$

Hence greatest prime dividing 12260 leaving remainder 17 is 53.

8. Find the largest number which would divide 50 and 60 leaving remainders 8 and 4 respectively. 6

Soln.

50–8 = 42

60–4 = 56

Now 42 = 2×3×7

56 = 2×2×2×7

Hence HCF of 42 and 56 is 2×7=14.

Hence the largest number dividing 42 and 56 leaving remainders 8 and 4 respectively is 14.

9. A student was asked to divide a number by 385. But in stead of applying long division method he applied short division method by using factors of 385 viz 5, 7 and 11 and in the process he obtained remainders 2, 4 and 10 respectively. What would be the remainder if the method of long division by 385 is applied? 6

Soln.

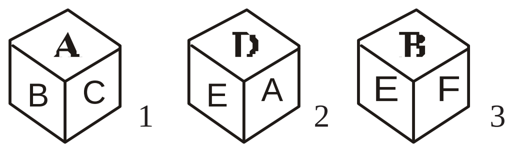
If q_1, q_2, q_3, be the quotients obtained by dividing the number N by 5, 7 and 11 leaving remainders 2, 4 and 10 respectively.

Then

$$N = 5q_1 + 2$$
$$q_1 = 7\times q_2+4 \quad \text{and} \quad q_2=11q_3+10$$
$$\therefore N = 5\,(7q_2+4) = 35\,q_2+20$$
$$= 35\,(11q_3+10)+20$$
$$= 385q_3+350+20$$
$$= 385q_3+370$$

The remainder obtained by dividing the number by 385 is 370.

10. Three different views of the same cube with differently coloured faces are shown below. What is the colour of the bottom face (the face opposite to A) in figure 1? 6



Soln.

From figure 2 and figure 3, four faces adjacent to E are A, D (in fig2), B and F (as in fig-3)

Therefor the face apposite to A in fig2 is B or F. But B is adjacent to A in fig-1.

Thus the face opposite to A must be F.

11. There is a circular path around a sports field. Priya, Neha and Mina respectively take 18 minutes, 12 minutes and 8 minutes to drive one round of the field. If they start together at the same point and along the same direction, after how many minutes will they meet again at the starting point? 6

Soln.

The time required by the three runners to come together for the first time after start must be the LCM of 18, 12 and 8

Now

$$\begin{array}{r|l} 6 & 18,\ 12,\ 8 \\ \hline 2 & 3,\ 2,\ 8 \\ \hline & 3,\ 1,\ 4 \end{array}$$

Hence required time of meeting together after start is $6\times2\times3\times4=144$ minutes.

12. At what time between 7 and 8 O'clock the hour hand and minute hand will be together? 7

Soln.

At 7, the hour hand is at 7 and minute hand is at 12.

Let the hour hand crosses x divisions from 7 when the minute hand overlaps with the hour hand. Then the minute hand has already crossed 35+x division. But the ratio of divisions crossed by hour hand and minute hand is 5:60 or 1:12

$$\therefore \frac{x}{1} = \frac{35 + x}{12}$$

$$\Rightarrow 12x - x = 35$$

$$\Rightarrow x = \frac{35}{11} = 3\frac{2}{11}$$

$\therefore$The hour hand and minute hand will come together between 7 and 8 hour at $35 + 3\frac{2}{11} = 38\frac{2}{11}$ minutes past 7 o'clock.

13. The cost of 4 chairs and 5 tables is Rs. 14800/- and that of 5 chairs and 4 tables is Rs. 14000/-. Find the price of a chair and of a table. 7

Soln.

Cost of 4 chairs and 5 tables is Rs. 14800
Cost of 5 chairs and 4 tables is Rs. 14000

Subtracting–

Cost of 1 table - Cost of 1 chair is 14800-14000=800
$\therefore$ Cost of 1 table = Rs. 800 + cost of 1 chair.
Then the cost of 4 chairs and 5 tables.
= cost of 4 chairs and cost of 5 chairs + 4000
= cost of 9 chairs + 4000

By condition,

Cost of 9 chairs + 4000 = 14800
$\therefore$Cost of 9 chairs = 10800
$\therefore$Cost 1 chair is 10800÷9
=1200
Therefore the cost of 1 table is 1200 + 800
= 2000.

14. Two trains 100 kilometers apart are moving at a speed of 10 and 15 kilometers per hour opposite to each other. If the slower train starts at 3 PM and the other starts at 2 PM, at what time will they meet together? 6

Soln.

The trains are 100 Km apart.
Let T_1 and T_2 be the trains moving at 10 Km and 15 Km per hour towards each other.
By condition T_1 starts at 3 PM while T_2 starts at 2 PM.
By the time T_1 starts moving, T_2 has already moved 15 Km towards T_1.
Therefore the trains are 100-15=85 km apart at 3 PM.
If T_1 and T_2 travel x and y km respectively when they meet together

Then $\frac{x}{10} = \frac{y}{15}$

$15x = 10y$
But $x+y = 85$
$\therefore 15x = 10(85-x)$
$\Rightarrow 15x+10x=850$

$$\Rightarrow x = \frac{850}{25} = 34$$

Then $y=85-34=51$

Hence the trains meet together after $\frac{34}{10}$ or $\frac{51}{15}$ hours after 3 PM.

In other words, the two trains will meet together in $3\frac{2}{5}$ hours after 3PM. i.e. at 6 hours 24 minutes PM.

15. Solve the following SUDOKU by inserting the numbers 1 through 9 in the blank squares such that each of these numbers appears only once in any row, column or any of the nine inner squares marked by bold lines. 10

	2	7		6			1	3
			2			9	5	4
3				8	1			6
		1			8	3	9	
	4						2	
	6	5	9			7		
6			7	1				9
7	1	8			4			
4	5			2		1	3	

Soln.

The SUDOKU is not in correct form. The correct form is –

5	2	7	4	6	9	8	1	3
1	8	6	2	7	3	9	5	4
3	9	4	5	8	1	2	7	6
2	7	1	6	4	8	3	9	5
9	4	3	1	5	7	6	2	8
8	6	5	9	3	2	7	4	1
6	3	2	7	1	5	4	8	9
7	1	8	3	9	4	5	6	2
4	5	9	8	2	6	1	3	7

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Methematical Talent Search Corner

Category-III : (For Classes- IX, X & X appeared)

Answer all the questions:

1. Find the 5000th term of the following sequence: 5

1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,...

Soln.

We observe that 1 appears once, 2 appears twice, 3 appears thrice, ..., n appears n times. Further, observe that 1st term is1, (1+2)th term is 2, (1+2+3)th term is 3. So, (1+2+3+.... + n)th term is n.

Thus, $\frac{n(n+1)}{2}$th term is n.

Similarly, [1+2+3+ .... + (n-1)] th terms is

n-1 i.e. $\frac{n(n-1)}{2}$th term is $n-1$..

So, all terms from $\left(\frac{n(n-1)}{2}+1\right)$th term to $\frac{n(n+1)}{2}$th term are equal to n.

We try to find the value of n for which

$$\frac{n(n-1)}{2} < 5000 \le \frac{n(n+1)}{2}$$

We see that $5000=50\times100 = \frac{100\times100}{2} < \frac{100\times101}{2}$

Also,

$$\frac{100\times99}{2} < \frac{100\times100}{2} = 5000 < \frac{100\times101}{2}$$

Thus, all terms from (50×99+1)th term to (50×101)the term are equal to 100.

$\therefore$The 5000th term is 100.

2. Let S = {(x,y,z) : $0\le x, y, z \le 9$ and x+y+z is divisible by 3}. Find the number of elements of the set S. 6

Soln.

Any number is either of the form 3k, or 3k+1 or 3k + 2. i.e. any numbers leaves remainder 0 or 1 or 2 when divided by 3.

The numbers from 0 to 9 can be grouped into three categories acordingly. A={0,3,6,9}, B={1,4,7}, C={2,5,8}

If x,y,z $\in$A, then x+y+z is divisible by 3.

No. of choices of (x,y,z) in that case is 4×4×4 (Multiplication rule)

If x,y,z $\in$B, then x+y+z is divisible by 3.

$\therefore$ No. of choices = 3×3×3.

If x,y,z $\in$ C then x+y+z is divisible by 3

$\therefore$ No. of choices of (x,y,z) is 3×3×3.

The only other cases where x+y+z is divisible by 3 are those where each of x,y,z belong to different sets A,B,C. There are $\angle 3 = 6$ such cases. In each case, the no. of choices for (x,y,z) is 4×3×3.

$\therefore$The total no. of elements of the set S.

= 4×4×4+3×3×3+3×3×3+6×4×3×3

= 64+27+27+216

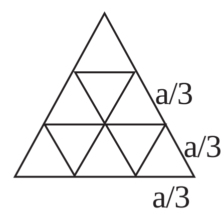
= 64+54+216

= 280+54

= 334

3. Show that out of any ten points chosen inside an equilateral triangle of side length a, there always exist two points whose distance apart is less than $\frac{a}{3}$ 6

Soln.



We trisect each side of the given equilateral triangle and join the points two at a time by line segments parallel to the side not containing the points. Thus, the whole area is divided into 9 smaller equilateral tringles each of side length a/3. Thus, marking ten points inside the triangle is equivalent to putting 10 objects in 9 boxes. Thus, one of the loopes will have two objects. This follows from pigeonhole principle. So, two points will be inside the same smaller triangle. Thus, their distance apart is less than a/3.

4. If $f:\mathbb{N}\to\mathbb{N}$ is a function satisfying $f(f(n)) + f(n+1)=n+2\ \forall n\in\mathbb{N}$, find the values of $f(1)$ and $f(2)$. 7

soln.

Taking n = 1,

$f(f(1)) + f(2)=3$

Since the codomain of f is $\mathbb{N}$, so the only possibilities are:

Case I: $f\{f(1)\}=1,\ f(2)=2$

Case II : $f\{f(1)\}=2,\ f(2)=1$

We have,

$f\{f(n)\}+f(n+1)=n+2$ –(*)

$$\Rightarrow f(n+1) = n+2-f\{f(n)\}\forall n \in \mathbb{N}$$

$$\Rightarrow f(n+1) \le n+2-1 \quad \{\because f(f(n)) \ge 1\}$$

$$\Rightarrow f(n+1) \le n+1 \quad \forall n \in \mathbb{N}$$

Similarly,

$f(f(n)) \le n+1\ \forall n\in\mathbb{N}$ ________ (2)

By (1), f(n) $\leq$ n $\forall n \geq 2$ ________ *(3)*
So for $n \geq 2$ and for $f(n) \geq 2$,
$f(f(n)) \leq f(n) \leq n$
$\Rightarrow f(f(n))-n \leq O$
$\Rightarrow 2-f(n+1) \leq O$
$\Rightarrow f(n+1) \geq 2$
Thus, if $n \geq 2$ and $f(n)) \geq 2$
then $f(n+1) \geq 2$.
Consider case I : $f(f(1))=1$, $f(2)=2$
By above,
$f(n) \geq 2 \ \forall \ n \geq 2$
Let $f(1) = c$
$\Rightarrow f(f(1)) = f(c)$
$\Rightarrow 1= f(c)$
$\Rightarrow c < 2 \quad (\because f(c) \geq 2$ if $c \geq 2)$
$\Rightarrow c = 1$
$\therefore f(1) = 1$.
Consider case II:
$f(f(1)) = 2$, $f(2) = 1$
Let $f(1) = c$
$\Rightarrow f(f(1)) = f(c)$
$\Rightarrow 2 = f(c)$
Putting n = 2 in (*)
$f(f(2)) + f(3) = 4$
$\Rightarrow f(1) + f(3) = 4$
$\Rightarrow f(3) = 4 - c$
$\because f: \mathbb{N} \Rightarrow \mathbb{N}$, so $f(3) \geq 1$
$\Rightarrow 4 - c \geq 1$
$\Rightarrow c \leq 3$
If c = 1, then $f(1)$ = C and 2 = $f(c)$ gives $f(1) = 1$ and $2 = f(1)$ which is not possible.
If c = 2, then $2 = f$ (c) gives $f(2) = 2$ but it contradicts $f(2) = 1$.
If c = 3, then $2 = f(c)$ gives
$f(3) = 2$ but $f(3) = 4 - c$
gives $f(3) = 4 - 3 = 1$.
Thus none of these are possible.
Hence, $f(1) = 1$, $f(2) = 2$.

5. Let $f(x)=x^3+ax^2+bx+c$ and $g(x)=x^3+bx^2+cx+a$, where *a b c* are integers with $c\neq0$. Suppose that the following conditions are satisfied:
(a) $f(1)=0$
(b) the roots of $g(x)=0$ are squares of the roots of $f(x)=0$.

Find the value of $a^{2019}+b^{2019}+c^{2019}$. 7

Soln.

$f(1)$-0

$\Rightarrow$ 1+a+b+c=0

$\Rightarrow$ a+b+c=-1

Let roots of $f(x)=0$ be α,β,γ

$\therefore$ Roots of $g(x)=0$ are α^2, β^2 γ^2

$f(x)=x^3+ax^2+bx+c=0$

$\Rightarrow$ Sum of roots =-a

$\Rightarrow \alpha + \beta + \gamma = -a$

Sum of products of roots taken 2 at a time = b

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = b$

Product of roots is -c

$\therefore \alpha\beta\gamma = -c$

Also,

$g(x) \equiv x^3 + bx^2 + cx + a = 0$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = -b$

$\Rightarrow (\alpha + \beta + \gamma)^2 -1 (\alpha\beta + \beta\gamma + \gamma\alpha) = -b$

$\Rightarrow a^2 - 2b = -b$

$\Rightarrow a^2 = b$

And

$\alpha^2\beta^2\gamma^2 = -a$

$\Rightarrow (\alpha\beta\gamma)^2 = -a$

$\Rightarrow (-c)^2 = -a$

$\Rightarrow c^2 = -a$

$\therefore$ a+b+c = -1

$\Rightarrow -c^2 + a^2 + c = -1 \Rightarrow -c^2 + c^4 + c = -1$

$\Rightarrow c^4 - c^2 + c + 1 = 0$

$\Rightarrow c^2(c^2-1) + (c+1) = 0$

$\Rightarrow c^2(c-1)(c+1) + (c+1) = 0$

$\Rightarrow (c+1)[c^2(c-1) +1] = 0$

$\Rightarrow (c+1)[c^3-c^2+1] = 0$

$\Rightarrow c = -1$ or $c^3 = c^2 - 1$.

But $c^3=c^2-1$. doesn't have integer solutions. If c is odd then c^2-1 is even. So c^3 is even but that is not correct as c is odd. If c is even, then c^2-1 is odd, so that c^3 is odd but that is not true as c is even.

Thus, c = -1

$\therefore a = -c^2 = -1$

$b = a^2 = (-1)^2 = 1$

$\therefore a^{2019} + b^{2019} + c^{2019}$

$= -1 + 1 - 1$
$= 1$

6. A positive integer has unit digit 6. If we erase this unit digit and place it in front of the remaining digits, we get 4 times the original number. Determine the smallest such positive integer. 7

Soln.

Let the integer be 10a+6 (n digits)
$10^{n-1} \times 6+a = 4\times(10a+6)$
$\Rightarrow (10^{n-1}-4) \times 6 = 39a$
$\Rightarrow (10^{n-1}-4) \times 2 = 13a$
$\therefore 13 | (10^{n-1}-4)$
So we need to find the smallest n for which $13|10^{n-1}-4$
i.e. $10^{n-1} \equiv 4 \bmod 13$
We have $10 \equiv -3 \bmod 13$
$\Rightarrow 10^2 = 9 \bmod 13$
$10^2 \equiv 9 \bmod 13$
$\Rightarrow (10^2)^2 \equiv 9^2 \bmod 13$
$\Rightarrow 10^4 \equiv 81 \bmod 13$
$\Rightarrow 10^4 \equiv 3 \bmod 13$
$\Rightarrow 10^5 \equiv 30 \bmod 13$
$\Rightarrow 10^5 \equiv 4 \bmod 13$
$\Rightarrow 10^{6-1} \equiv 4 \bmod 13$
$\therefore$ Least such n is n=6.
$\therefore 13a = 2 \times (10^{6-1}-4)$
$\Rightarrow 13a = 2 \times 99996$

$$\Rightarrow a = 2\times\frac{99996}{13}$$

$\Rightarrow a = 2 \times 7692$
$\Rightarrow a = 15384$
$\therefore$ The number is 153846

7. Write the number of perfect squares, perfect cubes and perfect fourth powers from 1 to 10^6 (both inclusive). How many of the numbers from 1 to 10^6 are neither perfect squares, nor perfect cubes nor perfect fourth powers? 3+5=8

Soln.

The perfect squares from 1 to 10^6 are $1^2, 2^2, 3^2, 4^2, \ldots, (10^3)^2$
So, there are 1000 perfect squares. The perfect cubes from 1 to 10^6 are $1^3, 2^3, 3^3, \ldots, (10^2)^3$
So, there are 100 perfect cubes. Clearly, the no. of perfect fourth powers will be less than 100.
$10^6 = 1000 \times 1000$

Perfect square nearest to 1000 is $31^2=961$.

$\therefore$ The fourth power nearest to 10^6

is $31^4 = 31^2\times31^2$

$= 961\times961$

$\therefore$ No. of perfect fourth powers upto 10^6 is 31.

(One mark for the 1st two and 2 marks for the fourth power)

Let A=Set of perfect squres from 1 to 10^6

B=Set of perfect cubes from 1 to 10^6

C=Set of perfect fourth powers from 1 to 10^6.

$\therefore$ n(A)=1000, n(B) = 100, n(c)=31.

$A \cap B$ is the set of perfect squares which are also perfect cubes & vice versa. This set contains elements which are cubes of perfect squares or equivalently squares of perfect cubes.

$\therefore$ The elements in $A \cap B$ are $(1^2)^3$, $(2^2)^3$, $(3^2)^3$, ..., $(102)^3$. Thus, n($A \cap B$) = 10.

The elements in $B \cap C$ are

$(1^3)^4$, $(2^3)^4$ & $(3^3)^4$ (upto $30)^4$

n($B \cap C$)=3.

The elements in $A \cap C$ are just the perfect fourth powers as every perfect fourth power is also a perfect square.

$\therefore$ n($A \cap C$) = 31

The numbers which are perfect squares as well as perfect cubes as well as perfect fourth powers are

1^{12}, 2^{12} and 3^{12} (i.e. power should be LCM of 2,3,4

$\therefore n(A \cap B \cap C) = 3$

$\therefore$ By inclusion-exclusion principle,

$n(A^c \cap B^c \cap C^c)$

$= n(S) - \{n(A) + n(B) + n(c)\} + \{n(A \cap B) + n(B \cap C) + n(A \cap C)\} - n(A \cap B \cap C)$

= 10^6-(1000+100+31)+(10+3+31)-3

= 1000000-1100+10

= 998900+10

= 998910

8. Let *a,b,c,* be the lengths of the sides *BC,CA* and *AB* of a triangle *ABC.* Consider all the possibilities:

(a) ABC is acute angled triangle

(b) A is an acute angle in a right angled triangle

(c) A is an acute angle in an obtuse angled triangle

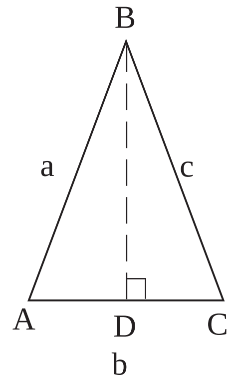
(d) A is an obtuse angle

(e) A is a right angle

In each case, prove that $a^2=b^2+c^2-2bc\cos A$. 2+1+2+2+1=8

Soln.

(a) ABC is acute angled $\triangle$



$BC^2=BD^2+DC^2$

$\Rightarrow a^2=AB^2-AD^2+DC^2$

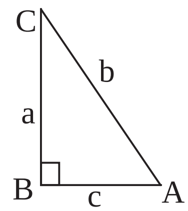
$\Rightarrow a^2=c^2-AD^2+(AC-AD)^2$

$\Rightarrow a^2=c^2-AD^2+AC^2-2AC.AD+AD^2$

$\Rightarrow a^2=c^2+AC^2-2.AC.AB\ \cos\ A$

$\Rightarrow a^2=c^2+b^2-2bc\ \cos\ A.$

(b) A is acute angle in a right angled $\triangle$



By Pythagoras theorm,

$a^2=b^2-c^2$

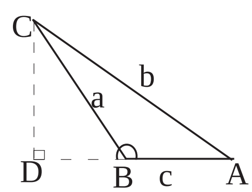
$\Rightarrow a^2=b^2+c^2-2c^2$

$\Rightarrow a^2=b^2+c^2-2.c.c$

$\Rightarrow a^2=b^2+c^2-2.(b\ \cos\ A)\ c.$

$\Rightarrow a^2=b^2+c^2-2bc\ \cos\ A.$

(c) A is an acute angle in an obtuse angled triangle.



In $\triangle$ BCD,

$a^2=CD^2+BD^2$

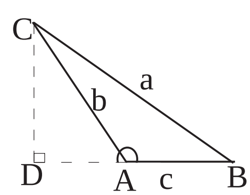
$\Rightarrow a^2=AC^2-AD^2+(AD-AB)^2$

$\Rightarrow a^2=b^2-AD^2+AD^2-2AD.AB+AB^2$

$\Rightarrow a^2=b^2+c^2-2.(AC\ \cos\ A).AB$

$\Rightarrow a^2=b^2+c^2-2bc\ \cos\ A.$

(d) A is an obtuse angle



In $\triangle$BCD,

$a^2=CD^2+BD^2$

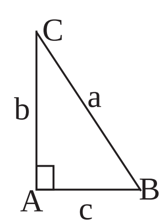
$\Rightarrow a^2=AC^2-AD^2+(BA+AD)^2$

$\Rightarrow a^2=b^2-AD^2+AB^2+AD^2+2AB.AD$

$\Rightarrow a^2=b^2+c^2+2.c.AC\ \cos\ (180-A)$

$\Rightarrow a^2=b^2+c^2-2bc\ \cos\ A.\ (\therefore \cos\ (180-A)=-\cos\ A)$

(e) A is a right angle.



By Pythogoras theorem

$a^2=b^2+c^2$

$\Rightarrow a^2=b^2+c^2-2bc.\cos\ 90$

$\Rightarrow a^2=b^2+c^2-2bc\ \cos\ A.$

9. A circle has centre on the side AB of a cyclic quadrilateral ABCD. The other three sides are tangents to the circle. Draw the diagram and prove that AD+BC=AB. 2+6=8

Soln.

(2 marks for diagram)

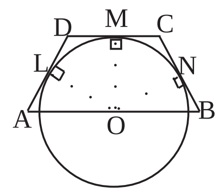
Since ABCD is cyclic.

So,

$\angle A + \angle C = 180^0$

$\angle B + \angle D = 180^0$

Let OL, OM and ON be the radii at the points of contact of the sides AD, DC and CB respectively. Then OL $\perp$ AD, OM $\perp$ DC, ON $\perp$ CB



Const : X and Y are marked on AD & BC such that AX=AO and BY=BO.

$$\therefore \angle AXO = \left(\frac{180^0 - A}{2}\right) = 90^0 - \frac{A}{2}$$

$$\& \angle BYO = 900 - \frac{B}{2}$$

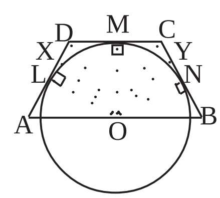
In $\triangle$OLX and $\triangle$OCM

$\angle LXO = \angle MCO = 90^0 - \frac{A}{2}$

$\angle OLX = \angle OMC$ (Each 90^0)

OL=OM(radii)

$\triangle OLX \cong \triangle OCM$



(AAS congruency)

$\therefore$ LX=MC

But MC=CN (tangents from C)

LX=CN

Similarly,

NY=DL

$\therefore$ AB=AO+OB

=AX + BY

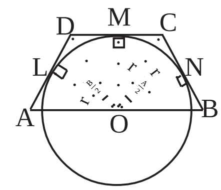
=AL+LX+BN+NY

=AL+CN+BN+DL

=(AL+DL)+(CN+BN)

=AD+BC

Another proof using trigonometry



$$\angle LOM = 180^0 - \angle D$$
$$= \angle B$$

$$\therefore \angle LOD = \angle DOM = \frac{\angle B}{2}$$

$$\angle NOC = \angle COM = \frac{\angle A}{2}$$

AD+BC=AL+LD+BN+NC

$$= r\left[\tan(90^0 - A) + \tan\frac{B}{2} + \tan(90^0 - B) + \tan\frac{A}{2}\right]$$

$$= r\left[\cot A + \tan\frac{B}{2} + \cot B + \tan\frac{A}{2}\right]$$

$$= r\left[\frac{1}{\tan A} + \tan\frac{A}{2} + \tan\frac{B}{2} + \frac{1}{\tan B}\right]$$

$$= r\left[\frac{1-\tan^2\frac{A}{2}}{2\tan\frac{A}{2}} + \tan\frac{A}{2} + \tan\frac{B}{2} + \frac{1-\tan^2\frac{B}{2}}{2\tan\frac{B}{2}}\right]$$

$$= r\left[\frac{1+\tan^2\frac{A}{2}}{2\tan\frac{A}{2}} + \frac{1+\tan^2\frac{B}{2}}{2\tan\frac{B}{2}}\right]$$

$$= r\left[\frac{1}{\operatorname{Sin} A} + \frac{1}{\operatorname{Sin} B}\right]$$

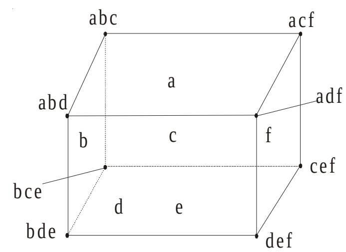
$= r[\operatorname{Cosec} A + \operatorname{cosec} B]$

$= AO + OB$

$= AB$

10. Positive integers *a,b,c,d,e,f* are written on the six faces of a cube, one on each. At each of the eight corners (vertices), the product of the numbers on the faces that meet at that corner is written. The sum of the numbers written on the corners is 4444. Find all possible values of the sum of the numbers written on the faces. 8

Soln.



Sum of numbers written on the corners = abc+abd+acf+adf+bce+bde+def+cef

= ab(c+d)+af(c+d)+be(c+d)+cf(c+d)

= (c+d) [a(b+f)+e(b+f)]

= (c+d) (a+c) (b+f)

= 4444

= 4×1111

= 4×11×101

Since the intigers are positive, so each of the sums c+d, a+e and b+f must be greater than 1.

So, there are the following possibilities:

(c+d) (a+e) (b+f) = 4×11×101

So, (c+d)+(a+e)+(b+f) = 4+11+101

= 116

(c+d) (a+e) (b+f) = 2×22×+101

∴ (c+d)+(a+e)+(b+f) = 2+22+101

= 125

(c+d) (a+e)(b+f)=2×11×202

∴ (c+d)+(a+e)+(b+f)=2+11+202

= 215

∴ (c+d)+(a+e)+(b+f)=2×2×1111

∴ (c+d)+(a+e)+(b+f) = 2+2+1111

= 1115

There are four possible values

11. Let *x,y,z* be non-negative real numbers such that *x+y+z*=1, prove that

$$0 \le xy+yz+zx-2xyz \le \frac{7}{27}.$$

(A hint : Put $x=a+\frac{1}{3}$, $y=b+\frac{1}{3}$, $z=c+\frac{1}{3}$ with appropriate restrictions on a, b and c.) 10

Soln.

Let $x = a+\frac{1}{3}, y = b+\frac{1}{3}, z = c+\frac{1}{3}$

∴ x+y+z=1

=> a+b+c=0
Since x,y,z are non-negative,

So $a \ge -\frac{1}{3}, b \ge -\frac{1}{3}, c \ge -\frac{1}{3}$

xy+yz+zx-2xyz

$$=\left(a+\frac{1}{3}\right)\left(b+\frac{1}{3}\right)+\left(b+\frac{1}{3}\right)\left(c+\frac{1}{3}\right)+\left(c+\frac{1}{3}\right)\left(a+\frac{1}{3}\right)-2\left(a+\frac{1}{3}\right)\left(b+\frac{1}{3}\right)\left(c+\frac{1}{3}\right)$$

$$\Rightarrow xy+yz+zx-2xyz$$

$$=ab+\frac{a+b}{3}+\frac{1}{9}+bc+\frac{b+c}{3}+\frac{1}{9}+ca+\frac{a+c}{3}+\frac{1}{9}-2\left(a+\frac{1}{3}\right)\left(bc+\frac{b+c}{3}+\frac{1}{9}\right)$$

$$=(ab+bc+ca)+\frac{2(a+b+c)}{3}+\frac{3}{9}$$

$$-2\left[abc+\frac{ab+ac}{3}+\frac{a}{9}+\frac{bc}{3}+\frac{b+c}{9}+\frac{1}{27}\right]$$

$$=(ab+bc+ca)+\frac{2}{3}\times 0+\frac{1}{3}-2abc-\left(\frac{ab+bc+ca}{3}\right)-\frac{1}{27}$$

$$=\frac{2}{3}(ab+bc+ca)-2abc+\frac{7}{27}$$

$$=\frac{2}{3}[ab+bc+ca-3abc]+\frac{7}{27}$$

$$=\frac{2}{3}\left[-\frac{1}{2}(a^2+b^2+c^2)-(a^3+b^3+c^3)\right]+\frac{7}{27}$$

$(\because a+b+c=0)$

$$=-\frac{1}{3}\left[a^2+b^2+c^2+2a^3+2b^3+2c^3\right]+\frac{7}{27}\le\frac{7}{27}$$

as 1+2a, 1+2b, 1+2c $\ge$0

$(\because a,b,c \ge -\frac{1}{3})$

For the other part, we have xy+yz+zx-2xyz
=xy(1-z)+yz(1-x)+zx
=xy(x+y)+yz(y+z)+zx$\ge$0

12. By trial & error or otherwise, find four different solutions (a, b, c, n) in positive integers of the equation 2^n = a! + b! + c!. Justify that these are the only possible solutions.

4+6=10

Soln.

One solution is a=1, b=1, c=2

$$\therefore \angle a + \angle b + \angle c = 1 + 1 + 2 = 4 = 2^2$$

i.e $n = 2$.

Observe that of a,b,c $\geq$3, then there exist no solutions as in that case,

$$\underline{|3}\left(\frac{\underline{|a}}{\underline{|3}} + \frac{\underline{|b}}{\underline{|3}} + \frac{\underline{|c}}{\underline{|3}}\right) - 2^n$$

i.e. $3|2^n$ but that is not possible.
So, all three of a,b,c, cannot be greater than or equal to 3.
Again if a=b=c then $3|2^n$ which is not possible. So, a,b,c, cannot be all equal. without loss of generality,

Let $a \leq b \leq c$. Then

$$2^n = \underline{|a}\left(1 + \frac{\underline{|b}}{\underline{|a}} + \frac{\underline{|c}}{\underline{|a}}\right)$$

$$\therefore \underline{|a} \mid 2^n$$

$$\Rightarrow a = 1 \text{ or } 2$$

$$\text{If } a = 1, \text{ then } 2^n - 1 = \underline{|b} + \underline{|c}$$

$$\Rightarrow 2^n - 1 = \underline{|b}\left(1 + \frac{\underline{|c}}{\underline{|b}}\right)$$

$\Rightarrow \underline{|b} \mid 2^n - 1$ but $2^n - 1$ is odd. So $\underline{|b}$ can divide 2^n-1 if and only if b=1.
So if a=1 then b=1
$\therefore$ 2^n=1+1+Lc
=> 2^n-2=Lc
=> $2(2^{n-1}-1)$ = Lc

Since 2^{n-1}-1 is odd So $2|\underline{|c}$ but $2^2 \nmid \underline{|c}$. Since c=2 or 3

If c=2, then

2^n=1+1+$\underline{|2}$

$\Rightarrow 2^n=4 \Rightarrow n=2$

$\therefore$ (1, 1, 2, 2) is a soln.

If c=3, then $2^n = 1+1+\lfloor\underline{3}$

$\Rightarrow 2^n=8$

$\Rightarrow n=3$

$\therefore$ (1, 1, 3, 3) is a soln.

Next if a=2, then $2^n=2+\lfloor\underline{b}+\lfloor\underline{c}$

$$\Rightarrow 2\times\left(2^{n-1}-1\right)=\lfloor\underline{b}+\lfloor\underline{c} \;=\lfloor\underline{b}\left(1+\frac{\lfloor\underline{c}}{\lfloor\underline{b}}\right)$$

So, $\lfloor\underline{b}\,|\,2(2^{n-1}-1)$

Since $2^{n-1}-1$ is odd, so b=1 or 2 or 3.

But since $a\le b\le c$, So $b\ne 1$.

If b=2, then $2^n=2+2+\lfloor\underline{c}$

$\Rightarrow 2^n-4=\lfloor\underline{c}$

$\Rightarrow 2^2(2^{n-2}-1)=\lfloor\underline{c}$

$\Rightarrow 2^2|\lfloor\underline{c}$ but $2^3|\lfloor\underline{c}$

$\therefore$ c=4

$\therefore$ $2^n=2+2+\lfloor\underline{4}$

$\Rightarrow 2^n=2+2+24$

$\Rightarrow 2^n=28$

Which has no solution in integers.

If b=3, then $2^n=2+6+\lfloor\underline{c}$

$\Rightarrow 2^n-8=\lfloor\underline{c}$

$\Rightarrow 2^3)2^{n-3}-1)=\lfloor\underline{c}$

$\Rightarrow 2^3|\lfloor\underline{c}$ but $2^4\nmid\lfloor\underline{c}$

$\therefore$ c=4 or c=5

If c=4 then $2^n=2+6+\lfloor\underline{4}$

$\Rightarrow 2^n=32$

$\Rightarrow n=5$

$\therefore$ (2, 3, 4, 5) is a soln.

If c=5, then $2^n=2+6+\lfloor\underline{5}$

$\Rightarrow 2^n=128$

$\Rightarrow n=7$

$\therefore$ (2, 3, 5, 7) is a soln.

Thus, the only possible solutions are (1, 1, 2, 2), (1, 1, 3, 3,), (2, 3, 4, 5) and (2, 3, 5, 7).

13. Let $3k+2$ be a prime number and a,b be positive integers such that

$$\frac{a}{b}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...-\frac{1}{2k}+\frac{1}{2k+1}$$

Show that 3k+2 divides a.

(Hint : Group the sum in RHS into positive and negative terms. Simplify and rearrange suitably to extract 3k+2 from the sum. Then use the difinition of prime number.)

Soln.

Since 3k+2 is prims, so K is odd.

$$\frac{a}{b}=\left(1+\frac{1}{3}+\frac{1}{5}+......+\frac{1}{2k+1}\right)-\left(\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2k}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{2k}+\frac{1}{2K+1}\right)-2\left(\frac{1}{2}+\frac{1}{4}+.....+\frac{1}{2k}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{2k}+\frac{1}{2K+1}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{k}\right)$$

$$=\frac{1}{K+1}+\frac{1}{K+2}+....+\frac{1}{2K}+\frac{1}{2K+1}$$

$$=\left(\frac{1}{K+1}+\frac{1}{2K+1}\right)+\left(\frac{1}{K+2}+\frac{1}{2K}\right)+....+\left(\frac{1}{K+\frac{K+1}{2}}+\frac{1}{K+\frac{K+1}{2}+1}\right)$$

($\therefore$K is odd, so K+1 is even)

$$=\frac{3K+2}{(K+1)(2K+1)}+\frac{3K+2}{(K+2)(2K}+....+\frac{3K+2}{\left(K+\frac{K+1}{2}\right)\left(K+\frac{K+1}{2}+1\right)}$$

$$=(3K+2)\left[\frac{P}{(K+1)(K+2).....(2K)\,(2K+1)}\right]$$ where P is a +ve integer.

$$\Rightarrow a\,(K+1)(K+2)....*\,(2K)(2K+1)=(3K+2)\text{p.b.}$$

$\therefore$ 3K+2|(LHS)

Since 3K+2 is prime, so 3K+2 divides one of the factors but none of the factors K+1, K+2, ..., 2K, 2K+1 is divisible by 3K+2, so 3K+2 divides a.

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