Titu's Lemma

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Introduction

In 2001, the well known Mathematical Olympiad expert *Titu Andrescu* delivered a lecture on a special case of *Cauchy-Schwarz inequality* (that inequality is actually *Bergstrom's inequality*) that could be effectively used in solving many inequality problems asked in various Mathematics Olympiads.

The technique was so useful in solving problems, that students often referred to it as *Titu's Lemma* and soon it just got the popular name "Titu's Lemma". In fact using *Titu's Lemma* even some old IMO inequality problems (like that of IMO 1995, IMO Shortlist 1996, etc), USAMO, and many other Mathematical Olympiads worldwide could be easily solved.

Titu's Lemma

If a and b are real numbers, and x and y are positive real numbers, then

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}.$$

Proof. We have

$$\begin{aligned}\frac{a^2}{x} + \frac{b^2}{y} - \frac{(a+b)^2}{x+y} &= \frac{a^2y(x+y) + b^2x(x+y) - xy(a^2+b^2+2ab)}{xy(x+y)} \\ &= \frac{(ay-bx)^2}{xy(x+y)} \geq 0.\end{aligned}$$

This proves the Lemma. □

Note: The equality occurs when $\frac{a}{x} = \frac{b}{y}$.

The Lemma can be generalized to any finite number of variables. For example

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b)^2}{x+y} + \frac{c^2}{z} \geq \frac{\{(a+b)+c\}^2}{(x+y)+z},$$

where a, b, c are real numbers and x, y, z are positive real numbers. Here the equality occurs when $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

This lemma is quite important in Maths Olympiads. To see how to apply this lemma, let us have a look at some direct questions first.

Solved Example

Example 1. Let x, y, z be positive real numbers. Prove that

$$\frac{y^2+z^2}{x} + \frac{z^2+x^2}{y} + \frac{x^2+y^2}{z} \geq 2(x+y+z).$$

(RMO, 2014)

Solution. We have

$$\begin{aligned}\frac{y^2+z^2}{x} + \frac{z^2+x^2}{y} + \frac{x^2+y^2}{z} &= \left(\frac{y^2}{x} + \frac{z^2}{y} + \frac{x^2}{z}\right) + \left(\frac{z^2}{x} + \frac{x^2}{y} + \frac{y^2}{z}\right) \\ &\geq \frac{(y+z+x)^2}{x+y+z} + \frac{(z+x+y)^2}{x+y+z} \\ &= 2(x+y+z).\end{aligned}$$

■

Example 2. Let x, y, z be positive real numbers satisfying $x + y + z = 1$. Prove that

$$xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2 \geq 4xyz.$$

(Assam Maths Olympiad, 2014)

Solution. We have to prove that $\frac{(x+y)^2}{z} + \frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} \geq 4$.

Now, $\frac{(x+y)^2}{z} + \frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} \geq \frac{(x+y+y+z+z+x)^2}{z+x+y}$. ■

Example 3. For a, b, c, d positive real numbers, prove that $\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$.

(South Africa, 1995)

Solution. We have $\frac{1^2}{a} + \frac{1^2}{b} + \frac{2^2}{c} + \frac{4^2}{d} \geq \frac{(1+1+2+4)^2}{a+b+c+d}$. ■

Example 4. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

(Belarus, 1999)

Solution. We have

$$\frac{1^2}{1+ab} + \frac{1^2}{1+bc} + \frac{1^2}{1+ca} \geq \frac{9}{3+ab+bc+ca}.$$

Since $ab + bc + ca \leq a^2 + b^2 + c^2$, so we will get the required result. ■

Note: $\dfrac{a^2}{1}+\dfrac{b^2}{1}+\dfrac{c^2}{1} \geq \dfrac{(a+b+c)^2}{1+1+1}$ implies $ab+bc+ca \leq a^2+b^2+c^2$.

Example 5. For any real number $x, y > 1$ prove that $\dfrac{x^2}{y-1}+\dfrac{y^2}{x-1} \geq 8$.

(Russia, 1992)

Solution. We have

$$\begin{aligned}\frac{x^2}{y-1}+\frac{y^2}{x-1} &\geq \frac{(x+y)^2}{(y-1)+(x-1)}\\ &= \frac{\{(x+y-2)+2\}^2}{x+y-2}\\ &= \{(x+y-2)+\frac{4}{(x+y-2)}\}+4\\ &\geq 2\sqrt{(x+y-2)\frac{4}{(x+y-2)}}+4\\ &= 8.\end{aligned}$$

■

Example 6. If the positive real number a, b, c satisfy $a^2+b^2+c^2=3$, prove that

$$\frac{a^2}{2+b+c^2}+\frac{b^2}{2+c+a^2}+\frac{c^2}{2+a+b^2} \geq \frac{(a+b+c)^2}{12}.$$

(Baltic Way, 2008)

Solution. We have

$$\begin{aligned}\frac{a^2}{2+b+c^2}+\frac{b^2}{2+c+a^2}+\frac{c^2}{2+a+b^2} &\geq \frac{(a+b+c)^2}{6+(a+b+c)+(a^2+b^2+c^2)}\\ &= \frac{(a+b+c)^2}{9+(a+b+c)}.\end{aligned}$$

So, now we need to prove that $a+b+c \leq 3$. Which is implied by $\dfrac{a^2}{1}+\dfrac{b^2}{1}+\dfrac{c^2}{1} \geq \dfrac{(a+b+c)^2}{1+1+1}$. ■

Example 7. Prove that the inequality $\dfrac{a^2}{(a+b)(a+c)}+\dfrac{b^2}{(b+c)(b+a)}+\dfrac{c^2}{(c+a)(c+b)} \geq \dfrac{3}{4}$ holds for all positive real number a, b, c.

(Croatia, 2004)

Solution. We have

$$\begin{aligned}
&\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}\\
&=\frac{a^2}{a^2+(ab+bc+ca)}+\frac{b^2}{b^2+(ab+bc+ca)}+\frac{c^2}{c^2+(ab+bc+ca)}\\
&\geq\frac{(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}\\
&=\frac{(a+b+c)^2}{(a+b+c)^2+(ab+bc+ca)}\\
&=\frac{1}{1+\frac{ab+bc+ca}{(a+b+c)^2}}.
\end{aligned}$$

So, we just need to prove that $\dfrac{ab+bc+ca}{(a+b+c)^2}\leq\dfrac{1}{3}$, which follows from $ab+bc+ca\leq a^2+b^2+c^2$. ■

Example 8. Let a, b, c be some positive numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

(Moscow, 1963; RMO, 1990)

Solution. We have

$$\begin{aligned}
\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}&=\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ca+cb}\\
&\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\\
&\geq\frac{3(ab+bc+ca)}{2(ab+bc+ca)},\ \textit{since}\ (a+b+c)^2\geq 3(ab+bc+ca)\\
&=\frac{3}{2}.
\end{aligned}$$

■

Note: This inequality is also called *Nesbitt's inequality.*

Example 9. If a and b are positive and $a+b=1$, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq\frac{25}{2}.$$

(INMO, 1988)

Solution.

$$\begin{aligned}
\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2&\geq\frac{\left(a+\frac{1}{a}+b+\frac{1}{b}\right)^2}{1+1}\\
&=\frac{\left(1+\frac{1}{ab}\right)^2}{2}\\
&\geq\frac{25}{2},
\end{aligned}$$

since $a+b=1$ implies $ab\leq\frac{1}{4}$. ■

Example 10. Prove that, for all positive real numbers a, b, c

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leq\frac{1}{abc}.$$

(USAMO, 1997)

Solution. We have to prove that

$$\frac{abc}{a^3+b^3+abc}+\frac{abc}{b^3+c^3+abc}+\frac{abc}{c^3+a^3+abc}\leq 1.$$

Now

$$\begin{aligned}\frac{abc}{a^3+b^3+abc}+\frac{abc}{b^3+c^3+abc}+\frac{abc}{c^3+a^3+abc}&=\frac{1}{\left(\frac{a^2}{bc}+\frac{b^2}{ca}\right)+1}+\frac{1}{\left(\frac{b^2}{ca}+\frac{c^2}{ab}\right)+1}+\frac{1}{\left(\frac{c^2}{ab}+\frac{a^2}{bc}\right)+1}\\&\leq\frac{1}{\frac{(a+b)^2}{c(b+a)}+1}+\frac{1}{\frac{(b+c)^2}{a(c+b)}+1}+\frac{1}{\frac{(c+a)^2}{b(a+c)}+1}\\&=\frac{c}{a+b+c}+\frac{a}{a+b+c}+\frac{b}{a+b+c}\\&=1.\end{aligned}$$

■

Example 11. If a, b, c are Positive real numbers and $a+b+c=1$, prove that

$$\frac{7+2b}{1+a}+\frac{7+2c}{1+b}+\frac{7+2a}{1+c}\geq\frac{69}{4}.$$

(Azerbaijan JBMO TST, 2015)

Solution. We have

$$\begin{aligned}\frac{7+2b}{1+a}+\frac{7+2c}{1+b}+\frac{7+2a}{1+c}&=5\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)+2\left(\frac{1+b}{1+a}+\frac{1+c}{1+b}+\frac{1+a}{1+c}\right)\\&\geq 5\frac{(1+1+1)^2}{3+a+b+c}+6\sqrt[3]{\frac{1+b}{1+a}.\frac{1+c}{1+b}.\frac{1+a}{1+c}}\\&=\frac{45}{3+1}+6\\&=\frac{69}{4}.\end{aligned}$$

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Example 12. Determine the minimum value of the expression:

$$\frac{a+1}{a(a+2)}+\frac{b+1}{b(b+2)}+\frac{c+1}{c(c+2)},$$

for positive real numbers a, b, c such that $a+b+c\leq 3$.

(Bosnia & Herzegovina TST, 2015)

Solution. We have

$$\begin{aligned}\frac{a+1}{a(a+2)}+\frac{b+1}{b(b+2)}+\frac{c+1}{c(c+2)}&=\frac{1}{2}\left[\frac{a+a+2}{a(a+2)}+\frac{b+b+2}{b(b+2)}+\frac{c+c+2}{c(c+2)}\right]\\&=\frac{1}{2}\left[\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\right)\right]\\&\geq\frac{1}{2}\left[\frac{(1+1+1)^2}{a+b+c}+\frac{(1+1+1)^2}{a+b+c+6}\right]\\&\geq\frac{9}{2}\left[\frac{1}{3}+\frac{1}{3+6}\right],\ \textit{since}\ a+b+c\leq 3\\&=2.\end{aligned}$$

■

Example 13. For all positive real numbers a, b, c satisfying $a+b+c=1$, prove that

$$\frac{a^4+5b^4}{a(a+2b)}+\frac{b^4+5c^4}{b(b+2c)}+\frac{c^4+5a^4}{c(c+2a)}\geq 1-ab-bc-ca.$$

(Turkey JBMO TST, 2013)

Solution. We have

$$\begin{aligned}&\frac{a^4+5b^4}{a(a+2b)}+\frac{b^4+5c^4}{b(b+2c)}+\frac{c^4+5a^4}{c(c+2a)}\\&=\left[\frac{a^4}{a^2+2ab}+\frac{b^4}{b^2+2bc}+\frac{c^4}{c^2+2ca}\right]+5\left[\frac{b^4}{a^2+2ab}+\frac{c^4}{b^2+2bc}+\frac{a^4}{c^2+2ca}\right]\\&\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+2(ab+bc+ca)}+5\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+2(ab+bc+ca)}\\&=6\frac{(a^2+b^2+c^2)^2}{(a+b+c)^2}\\&\geq 2(a^2+b^2+c^2),\ \textit{since}\ \frac{a^2}{1}+\frac{b^2}{1}+\frac{c^2}{1}\geq\frac{(a+b+c)^2}{1+1+1}\\&=1-\left[(a+b+c)^2-2(a^2+b^2+c^2)\right]\\&=1-(ab+bc+ca)-\left[(ab+bc+ca)-(a^2+b^2+c^2)\right]\\&\geq 1-ab-bc-ca,\ \textit{since}\ a^2+b^2+c^2\geq ab+bc+ca.\end{aligned}$$

■

Example 14. For all real numbers a, prove that $3(a^4+a^2+1)\geq(a^2+a+1)^2$.

(Kosovo, 2013)

Solution. We have

$$\frac{a^4}{1}+\frac{a^2}{1}+\frac{1}{1}\geq\frac{(a^2+a+1)^2}{1+1+1}.$$

■

Example 15. Let $x, y, z > 0$. Show that:

$$\frac{x^3}{z^3 + x^2y} + \frac{y^3}{x^3 + y^2z} + \frac{z^3}{y^3 + z^2x} \geq \frac{3}{2}.$$

(Romania JBMO TST, 2015)

Solution. Let $x^3 = a, y^3 = b, z^3 = c$. Then we get

$$\begin{aligned}
\frac{x^3}{z^3 + x^2y} + \frac{y^3}{x^3 + y^2z} + \frac{z^3}{y^3 + z^2x} &= \frac{a}{c + \sqrt[3]{a^2b}} + \frac{b}{a + \sqrt[3]{b^2c}} + \frac{c}{b + \sqrt[3]{c^2a}} \\
&\geq \frac{a}{c + \frac{a+a+b}{3}} + \frac{b}{a + \frac{b+b+c}{3}} + \frac{c}{b + \frac{c+c+a}{3}} \\
&= 3\left[\frac{a}{3c + 2a + b} + \frac{b}{3a + 2b + c} + \frac{c}{3b + 2c + a}\right] \\
&= 3\left[\frac{a^2}{3ca + 2a^2 + ab} + \frac{b^2}{3ab + 2b^2 + bc} + \frac{c^2}{3bc + 2c^2 + ca}\right] \\
&\geq 3\frac{(a + b + c)^2}{2(a^2 + b^2 + c^2) + 4(ab + bc + ca)} \\
&= \frac{3}{2}.
\end{aligned}$$

■

Exercise

Exercise 16. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1.$$

(IMO shortlist, 1996)

Exercise 17. If a, b, c are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

(RMO, 2006)

Exercise 18. Given real numbers $a, b, c, d, e \geq 1$, prove that

$$\frac{a^2}{c - 1} + \frac{b^2}{d - 1} + \frac{c^2}{e - 1} + \frac{d^2}{a - 1} + \frac{e^2}{b - 1} \geq 20.$$

(RMO, 2012)

(Hint: see solved Example 5.)

Exercise 19. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}.$$

(IMO, 1995)

(Hint: $\dfrac{1}{a^3(b+c)} = \dfrac{\left(\frac{1}{a}\right)^2}{ab+ac}$, etc.)

Exercise 20. Prove that $\dfrac{1+x^2}{1+y+z^2} + \dfrac{1+y^2}{1+z+x^2} + \dfrac{1+z^2}{1+x+y^2} \geq 2$, for any real numbers $x, y, z \geq 1$.

(JMBO, 2003)

(Hint: $x \leq \dfrac{1+x^2}{2}$, so $\dfrac{1+z^2}{1+x+y^2} \leq \dfrac{2(z^2+1)}{2(1+y^2)+1+x^2}$ etc. Then put $1+x^2 = a$ etc. to use Titu's Lemma.)

Exercise 21. Let a, b, c, d be positive real numbers with $a+b+c+d = 1$. Prove that $\dfrac{a^2}{a+b} + \dfrac{b^2}{b+c} + \dfrac{c^2}{c+d} + \dfrac{d^2}{d+a} \geq \dfrac{1}{2}$.

(Ireland, 1999)

Exercise 22. Let w, x, y, z be positive real numbers, prove that

$$\frac{w}{x+2y+3z} + \frac{x}{y+2z+3w} + \frac{y}{z+2w+3x} + \frac{z}{w+2x+3y} \geq \frac{2}{3}.$$

(Moldova, 2007)

Exercise 23. Let $a_1, a_2, \cdots a_n$ and $b_1, b_2, \cdots b_n$ be positive numbers with $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n$. Prove that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{1}{2}(a_1+a_2+\cdots+a_n).$$

(ADMO, 1991)

Exercise 24. Let $x_0 > x_1 > x_2 > \cdots > x_n$ be real numbers. Prove that

$$x_0 + \frac{1}{x_0-x_1} + \frac{1}{x_1-x_2} + \cdots + \frac{1}{x_{n-1}-x_n} \geq x_n + 2n.$$

(St. Petersburg, 1999)

(Hint: $(x_0-x_1)+(x_1-x_2)+\cdots+(x_{n-1}-x_n) = x_0-x_n$.)

Exercise 25. Prove that if $a_1, a_2, \cdots a_n$ are positive numbers whose sum is 1, then

$$\frac{a_1}{2-a_1} + \frac{a_2}{2-a_2} + \cdots + \frac{a_n}{2-a_n} \geq \frac{n}{2n-1}.$$

(Balkan, 1984)

(Hint: $\dfrac{a_i}{2-a_i} = -1 + 2\left(\dfrac{1^2}{2-a_i}\right)$.)