Diophantine Approximation and Its Importance

Abhishek Sarma

Numbers have always fascinated human beings. The world of science would certainly not be able to do anything without numbers. So it is very important for us to understand them. The first numbers known to humans were the natural numbers, that is, the numbers 1, 2, 3, ... As Leopold Kronecker famously said "God created the natural numbers. All else is the work of man." In this article, we will learn about understanding real numbers and their behaviour.

The reason we are interested in it is because we want to understand how real numbers behave, especially the irrational numbers. We know that an irrational number is a real number that cannot be expressed in the form $\frac{p}{q}$, where p and q are integers. The properties of natural numbers, integers, rational numbers are easy to be understood, but the problem lies in the understanding of irrational numbers. These are the numbers that are non terminating and non recurring, i.e., their decimal expansion never ends and that too without any pattern. In these circumstances it is really difficult to predict the behaviour of irrational numbers. So, mathematicians tried to understand them by approximating them with the help of rational numbers. The study of approximating real numbers with the help of rational numbers is called Diophantine approximation. Also, many important ideas in Number Theory stem from notions of Diophantine approximation.

We begin by defining the notion of a simple continued fraction expansion.

Definition 1. A simple continued fraction expansion for a real number a is given by:

$$a = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where n is a nonnegative integer and a_0 is an integer and a_i is positive for $i = 1, 2, 3, \dots, n$.

The numbers $a_0, a_1, \ldots, a_n$ are called partial denominators. For simplicity, conventionally, the simple continued fraction above is denoted by $[a_0; a_1, \ldots, a_n]$. The simple continued fraction made from $[a_0; a_1, \ldots, a_n]$ by cutting off the expansion after the kth partial denominator a_k is called the kth convergent of the given simple continued fraction and is denoted by C_k; in symbols, $C_k = [a_0; a_1, \ldots, a_k]$.

Convergents are very mportant in the theory of Diophantine approximation. This is because covergents are the closest approximations to any real number. This is explained in the following theorem:

Theorem 2. *For any rational number* $\frac{a}{b}$ *such that* $1 \leq b \leq q_k$*, we have*

$$\left| x - \frac{p_k}{q_k} \right| \leq \left| x - \frac{a}{b} \right|.$$

Equality in the last relation holds if $p_k = a$ *and* $q_k = b$*.*

This helps us to find targets for making better approximations of a real number. We'll take a look at how this works with the help of a few examples.

Example. Let us take the example of π. The simple continued fraction for π is given by [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, ...], with 3, $\frac{22}{7}, \frac{333}{106}, \frac{355}{113}$, and $\frac{103993}{33102}$ being the first few convergents. For a better understanding let us write π as

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{\ddots}}}.$$

Clearly, the first convergent is 3. The second convergent is calculated as follows: We cut down the expression at the point where 7 is the denominator. So, the second convergent is $3 + \frac{1}{7} = \frac{21+1}{7} = \frac{22}{7}$. Similarly, other convergents can be calculated.

Next we observe the following: Consider the convergent $\frac{22}{7}$ of π. From the above theorem, We have

$$\left| x - \frac{22}{7} \right| \leq \left| x - \frac{a}{b} \right|.$$

for any $a, b \in \mathbb{Z}$. That is, the fraction $\frac{22}{7}$ is closer to π than any other rational number whose denominator b is such that $1 \leq b \leq 7$. A natural question now is what happens when the denominator b of $\frac{a}{b}$ is greater than 7. In that case we choose the next convergent of π, i.e., $\frac{333}{106}$ and now this is a better approximation of π than $\frac{22}{7}$ as now it is closer to any other rational number whose denominator b is such that $1 \leq b \leq 106$. From here we can understand that as we go searching for more convergents we end up with better approximations of π. For other irrational numbers too, the same idea can be applied to approximate them.

Another interesting theorem is the Dirichlet approximation theorem. The theorem is as follows:

Theorem 3. *(Dirichlet approximation theorem). Let $x \in \mathbb{R}$ and let Q be a real number exceeding 1. Then there exist integers p and q with $1 \leq q < Q$ and $(p,\ q) = 1$ such that*

$$|qx - p| \leq \frac{1}{Q}.$$

Proof. This proof is a beautiful application of the box theorem or the pigeonhole principle. Write N = [Q], and consider the N + 1 real numbers

$$0, 1, \{x\}, \{2\}, \cdots, \{(N-1)x\},$$

where {x} is the fractional part of x. These N + 1 real numbers all lie in the interval [0, 1]. But if we divide this unit interval into N disjoint intervals of length 1/N, it follows that there must be two numbers from the above set which necessarily lie in the same interval. The difference between these two real numbers has the form $qx - p$ where p and q are integers such that $1 < q < N$ Thus we deduce that there exist integers p and q with $1 \leq q < Q$ such that $|qx - p| \leq \frac{1}{Q}$. The reason (p, q) = 1, is because we want $\frac{p}{q}$ in lowest terms. Otherwise, the coprimality can be seen if we divide both sides of the inequality by (p, q).

Corollary. For any irrational number x, there exist infinitely many pairs of integers p, q for which

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

From the above corollary, we can see that if we want to approximate an irrational number with the help of a rational number within a distance of $\frac{1}{100}$, then we have to make sure that the convergent we are choosing should not have a denominator greater than 10.

As one can observe, the continued fraction expansion of π is infinite. A natural question that comes to our mind is whether this will happen for all irrational numbers! The answer is indeed true. Another question that might come to our mind is whether the continued fraction expansion of rational numbers is finite! The situation in this case is that, in case of rational numbers the continued fraction expansion is finite. Which brings us to the following theorem:

Theorem 4. *A real number is rational if and only if the continued fraction expansion associated with it is finite and, a real number is an irrational number if and only if the continued fraction expansion associated with it is infinite.*

The case of an irrational number is π. Let us now understand what happens in the case of a rational number with the help of an example.

Example. Let us choose the rational number $\frac{15}{7}$. Now we can write

$$\frac{15}{7} = 2 + \frac{1}{7}.$$

This will happen with all the rational numbers. Now, as the example shows that all the rational numbers have a finite convergent, it is very easy to understand them because we can study the properties of all the convergents and understand their behaviour with the help of all the convergents. Unlike rational numbers, we cannot write all the convergents of irrational numbers. So, we can only understand them by getting close to them and the best possible way to do it is by taking the help of rational numbers whose properties are easier to understand.

The reader should now be in a better position to realise the importance of Diophantine approximation. It is advised to the reader take a look at the proofs of the theorems and the corollary which are not proved here. The reader is also advised to look at few more examples like e, $\frac{1+\sqrt{5}}{2}$ (The golden ratio), etc to understand the beauty of this topic. This will lead to various irrational numbers whose approximation is yet to done to a great extent.

"In a way, mathematics is the only infinite human activity. It is conceivable that humanity could eventually learn everything in physics or biology. But humanity certainly won't ever be able to find out everything in mathematics, because the subject is infinite. Numbers themselves are infinite. That's why mathematics is really my only interest."

– **Paul Erdős**