An overview of research on inverse eigenvalue problems for matrices described by graphs

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Research in any subject involves a good amount of study and patience and mathematics is no different. Sometimes it may happen that there is a considerable difference between what we aspire to do and what we actually end up doing. In the beginning of my PhD work back in 2012 at NIT Silchar, we had a strong intention to work on the spectral theory of unbounded linear operators in Hilbert spaces. However, we were not getting any lead even after six months of study and gradually we shifted our focus to eigenvalue theory of matrices. It felt quite tough as a lot of work was already available in the literature. Further survey into the literature brought us to a class of problems called inverse eigenvalue problems, thanks to the detailed survey by M. T. Chu [1].

The problem of reconstruction of a matrix having a desired structure and possessing a prescribed set of eigen data is known as an inverse eigenvalue problem, in short an IEP. The objective of an IEP is to construct matrices of a pre-assigned structure which satisfy the given constraints on eigenvalues and eigenvectors of the matrix or its submatrices. If there is no restriction on the structure of the required matrices, then an IEP becomes trivial. For example, it is very easy to find a matrix with eigenvalues $\{1, 2, \ldots, 100\}$. We can just take the diagonal matrix $\mathrm{diag}(1, 2, \ldots, 100)$. Thus, the level of difficulty of an IEP depends on the structure of the matrices that are to be reconstructed and on the type of eigen information available.

Special types of inverse eigenvalue problems were studied for various types of matrices like tridiagonal matrices, Jacobi matrices, arrow matrices, doubly arrow matrices, etc. by several authors ([2–6]). A well-known way of describing the structure of matrices is to represent them by graphs. An $n \times n$ symmetric matrix can be represented by an undirected graph on n vertices. So, we decided to include graphs in our research work on IEPs. We successfully worked on the reconstruction of special acyclic matrices like a path or a broom, from given eigen data [7, 8].

J. Peng, et. al. [6] and H. Pickman, et. al. [5, 9] studied IEPs involving the construction of arrow matrices and doubly arrow matrices from given eigen data consisting of the minimal and maximal eigenvalues of each of the leading principal submatrices of the required matrix. Motivated by this, we had studied the problem of constructing acyclic matrices whose graph is a broom from the same eigen data [8]. Recently, another paper of ours got published in the journal *Linear Algebra and its Applications* [10] where we studied an inverse eigenvalue problem, referred to as the minimax inverse eigenvalue problem, of constructing matrices whose graph is a special type of tree called a *generalized star of depth* 2 (Figure 1).

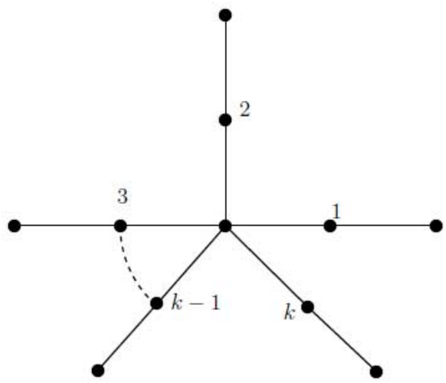

Figure 1: Generalized star of depth 2 (G_2S_k)

Let n and k be positive integers such that $n = 2k + 1$. Then, a *generalized star of depth* 2 on n vertices is obtained by attaching a pendant edge (that is, a path on two vertices) to each of the non-central vertices of a star on $k + 1$ vertices. A generalized star of depth 2 on $2k + 1$ vertices is denoted by G_2S_k.

For the convenience of the readers, let us recall a few concepts from graph theory. Let V be a finite set and let P be the set of all subsets of V that have only two distinct elements. Let $E \subset P$. Then $G = (V, E)$ is called a *graph* with vertex set V and edge set E. If $v_1, v_2 \in V$ and $\{v_1, v_2\} \in E$ then v_1v_2 is called an *edge* of G and the vertices v_1 and v_2 are said to be *adjacent*. The choice of P implies that the graphs under consideration are undirected and are free from multiple edges or loops. The degree of a vertex is the number of edges which are incident on it. A *pendant vertex* is a vertex of degree one. A sequence of distinct vertices $v_1, v_2, \ldots, v_n$ of G such that the consecutive vertices are adjacent is called a *path* of G. A graph is *connected* if there exists a path between every pair of its vertices. A *cycle* is a connected graph in which each vertex is adjacent to exactly two other vertices. A *tree* is a connected graph without any cycles.

Given an $n \times n$ symmetric matrix A, the graph of A, denoted by $G(A)$, has vertex set $\{1, 2, 3, \ldots, n\}$ and edge set $\{ij : i \neq j, a_{ij} \neq 0\}$. For a graph G with n vertices, $S(G)$ denotes the set of all $n \times n$ symmetric matrices that have G as their graph. An *acyclic* matrix is a matrix whose graph is a forest, that is, each of its connected components is a tree [11]. The $j \times j$ submatrix of a matrix A obtained from A by retaining only the first j rows and the first j columns of A is called the $j \times j$ leading principal submatrix of A.

We used a particular scheme of labelling the vertices of G_2S_k so as to express the corresponding matrices in a special form. We label the central vertex as 0 and the k vertices adjacent to it as

$1, 2, \ldots, k$. Let f be a permutation on the set $\{1, 2, \ldots, k\}$. Then we label the pendant vertex adjacent to the vertex $f(i)$ as $k+i$. This is illustrated for G_2S_3 in Figure 2 for all the $3! = 6$ possible ways of labelling. Under this scheme of labelling, the graph of each leading principal submatrix of the matrices of G_2S_k remains connected.

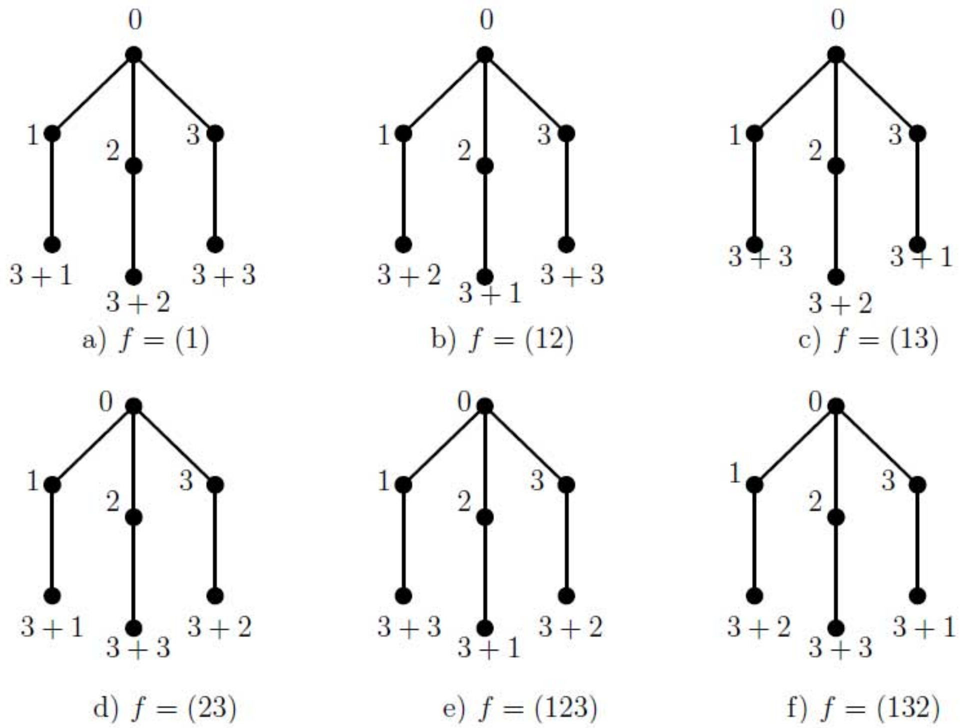

Figure 2: Labellings of G_2S_3

For any matrix $A_f \in S(G_2S_k)$, the diagonal entries are denoted by a_j where $0 \le j \le 2k$, the off-diagonal entry corresponding to the edge joining the central vertex to the vertex j, where $1 \le j \le k$, is denoted by b_j and the off-diagonal entry corresponding to the edge joining the vertex $f(j)$ with the pendant vertex $k+j$ is denoted by b_{k+j}. For example, any matrix of G_2S_3 labelled as in Figure 2(e) is written in the following form:

$$A_f = \begin{pmatrix} a_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ b_1 & a_1 & 0 & 0 & 0 & 0 & b_{3+3} \\ b_2 & 0 & a_2 & 0 & b_{3+1} & 0 & 0 \\ b_3 & 0 & 0 & a_3 & 0 & b_{3+2} & 0 \\ 0 & 0 & b_{3+1} & 0 & a_{3+1} & 0 & 0 \\ 0 & 0 & 0 & b_{3+2} & 0 & a_{3+2} & 0 \\ 0 & b_{3+3} & 0 & 0 & 0 & 0 & a_{3+3} \end{pmatrix}.$$

We studied the following minimax inverse eigenvalue problem:

IEPGS: Given $4k+1$ real numbers $\lambda_j, 1 \le j \le 2k+1$ and $\Lambda_j, 2 \le j \le 2k+1$ and a permutation

f on the set $\{1, 2, \ldots, k\}$, find a matrix $A_f \in S(G_2S_k)$ such that λ_j and Λ_j are respectively the minimal and maximal eigenvalues of the $A_{f,j}$, the $j \times j$ leading principal submatrix of A_f.

A sketch of the solution

The following results were used to find possible solutions of IEPGS:

Lemma 1. *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeros and λ_{min} and λ_{max} be the minimal and maximal zero of P respectively.*

- *If $\mu < \lambda_{min}$, then $(-1)^n P(\mu) > 0$.*
- *If $\mu > \lambda_{max}$, then $P(\mu) > 0$.*

Lemma 2. *If T is a tree then the minimal and maximal eigenvalues of any matrix $A \in S(T)$ are simple i.e. of multiplicity one. [Lemma 3.3 in [12]]*

The following lemma gives the relation among the leading principal minors of $\lambda I - A_f$:

Lemma 3. *Let $A_f \in S(G_2S_k)$. The sequence $\{P_j(\lambda) = det(\lambda I_j - A_{f,j})\}$ of characteristic polynomials of $A_{f,j}$ satisfies the following recurrence relations:*

(i) $P_1(\lambda) = \lambda - a_0$,

(ii) $P_{j+1}(\lambda) = (\lambda - a_j)P_j(\lambda) - b_j^2 \prod_{i=1}^{j-1} (\lambda - a_i)$ *for* $1 \leq j \leq k$,

(iii) $P_{k+j+1}(\lambda) = (\lambda - a_{k+j})P_{k+j}(\lambda) - b_{k+j}^2 Q_{f(j)}(\lambda)$ *for* $1 \leq j \leq k$,

where $Q_{f(j)}(\lambda)$ is the characteristic polynomial of the principal submatrix of $A_{f,k+j+1}$ obtained by deleting the rows and columns indexed by $f(j)$ and $k+j$.

The proof follows by expanding the determinant $det(\lambda I - A_{f,j})$ for each j where $2 \leq j \leq 2k+1$. By Cauchy's interlacing theorem [13, 14] , the eigenvalues of an $n \times n$ symmetric matrix and those of any of its $(n-1) \times (n-1)$ principal submatrices interlace each other. Hence, the given minimal and maximal eigenvalues of A_f satisfy the following inequalities

$$\lambda_{2k+1} \leq \lambda_{2k} \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \leq \Lambda_{2k} \leq \Lambda_{2k+1}. \tag{1}$$

As per the spectral constraint of IEPGS, λ_1 is an eigenvalue of $A_{f,1}$, so $P_1(\lambda_1) = 0$ from the first relation of Lemma 3, we have

$$a_0 = \lambda_1. \tag{2}$$

For $1 \leq j \leq k$, λ_{j+1} and Λ_{j+1} are respectively the minimal and maximal eigenvalues of $A_{f,j+1}$. So $P_{j+1}(\lambda_{j+1}) = 0$ and $P_{j+1}(\Lambda_{j+1}) = 0$, and from the second recurrence relation of Lemma 3, it follows that

$$\begin{aligned} a_j P_j(\lambda_{j+1}) + b_j^2 \prod_{i=1}^{j-1} (\lambda_{j+1} - a_i) - \lambda_{j+1} P_j(\lambda_{j+1}) = 0 \\ a_j P_j(\Lambda_{j+1}) + b_j^2 \prod_{i=1}^{j-1} (\Lambda_{j+1} - a_i) - \Lambda_{j+1} P_j(\Lambda_{j+1}) = 0. \end{aligned} \tag{3}$$

Note that (3) is a system of equations that are linear in a_j and b_j^2. We investigated the nature of solutions of the system. Let D_{j+1} denote the determinant of the coefficient matrix of the above system. Then,

$$D_{j+1} = P_j(\lambda_{j+1})\prod_{i=1}^{j-1}(\Lambda_{j+1} - a_i) - P_j(\Lambda_{j+1})\prod_{i=1}^{j-1}(\lambda_{j+1} - a_i). \tag{4}$$

Again for $1 \leq j \leq k$, λ_{k+j+1} and Λ_{k+j+1} are respectively the minimal and maximal eigenvalues of $A_{f,k+j+1}$. So $P_{k+j+1}(\lambda_{k+j+1}) = 0$ and $P_{k+j+1}(\Lambda_{k+j+1}) = 0$ and from the second recurrence relation of Lemma 3, it follows that

$$\begin{aligned} a_{k+j}P_{k+j}(\lambda_{k+j+1}) + b_{k+j}^2 Q_{f(j)}(\lambda_{k+j+1}) - \lambda_{k+j+1}P_{k+j}(\lambda_{k+j+1}) = 0 \text{ , and} \\ a_{k+j}P_{k+j}(\Lambda_{k+j+1}) + b_{k+j}^2 Q_{f(j)}(\Lambda_{k+j+1}) - \Lambda_{k+j+1}P_{k+j}(\Lambda_{k+j+1}) = 0, \end{aligned} \tag{5}$$

which is a system of linear equations in a_{k+j} and b_{k+j}^2. We investigated the nature of solutions of the system (5). Let D_{k+j+1} denote the determinant of the coefficient matrix of the above system. Then

$$D_{k+j+1} = P_{k+j}(\lambda_{k+j+1})Q_{f(j)}(\Lambda_{k+j+1}) - P_{k+j}(\Lambda_{k+j+1})Q_{f(j)}(\lambda_{k+j+1}). \tag{6}$$

The main result of our paper was the following theorem.

Theorem 4. *The IEPGS has a solution with unique values of $a_i, i = 0, 1, 2, \ldots, 2k$ and $b_i^2, i = 1, 2, \ldots, 2k$ if and only if*

$$\lambda_{2k+1} < \lambda_{2k} < \cdots < \lambda_2 < \lambda_1 < \Lambda_2 < \Lambda_3 < \cdots < \Lambda_{2k} < \Lambda_{2k+1}, \tag{7}$$

and the solution is given by the expressions (8) and (9) below

$$\begin{aligned} a_j &= \frac{\lambda_{j+1}P_j(\lambda_{j+1})\prod_{i=1}^{j-1}(\Lambda_{j+1} - a_i) - \Lambda_{j+1}P_j(\Lambda_{j+1})\prod_{i=1}^{j-1}(\lambda_{j+1} - a_i)}{D_{j+1}} \\ b_j^2 &= \frac{(\Lambda_{j+1} - \lambda_{j+1})P_j(\lambda_{j+1})P_j(\Lambda_{j+1})}{D_{j+1}}; \end{aligned} \tag{8}$$

$$\begin{aligned} a_{k+j} &= \frac{\lambda_{k+j+1}P_{k+j}(\lambda_{k+j+1})Q_{f(j)}(\Lambda_{k+j+1}) - \Lambda_{k+j+1}P_{k+j}(\Lambda_{k+j+1})Q_{f(j)}(\lambda_{k+j+1})}{D_{k+j+1}} \\ b_{k+j}^2 &= \frac{(\Lambda_{k+j+1} - \lambda_{k+j+1})P_{k+j}(\lambda_{k+j+1})P_{k+j}(\Lambda_{k+j+1})}{D_{k+j+1}}. \end{aligned} \tag{9}$$

Theorem 4 gives a general procedure of reconstructing all the $k!$ structures of matrices $A_f \in S(G_2S_k)$, under the modified scheme of labelling. Also, we are free to choose the signs of the off-diagonal entries as all the expressions and computations involve only the squares of the off-diagonal entries. Since there are $2k$ off-diagonal entries and each entry can be assigned either a plus sign or a minus sign, so for each permutation f we can reconstruct exactly 2^{2k} different matrices.

The paper was communicated to *Linear Algebra and its Applications* on 23rd February 2017. It took four years to complete the review process and was published on 16th March 2021. In between, it came back for major revisions a few times. It's a normal part of the review process and as researchers, we need to address the reviewers' queries properly. Suggestions from the reviewers also help in enhancing the quality of the research work. At the end, we conclude by insisting that there is no substitute to in-depth study, patience and perseverance in the world of research.

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