Acoustomicrofluidics: the wave equation & the acoustic streaming

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1. Introduction

Microfluidics is the study of science and technology that considers flow of fluids and suspensions of particles at a scale of the order of microns. Precise handling of fluids and suspension in a miniaturized and isolated environment have made the microfluidics more appealing for biological research [1]. For example, sorting and separation of microparticles in a continuous flow is required for chemical synthesis, mineral process, and biological applications [2]. Further, the mixing of reactants/reagents has paramount importance in biological processes, viz., drug delivery, enzyme reactions, protein synthesis. On the other hand, droplet-based microfluidics, which uses a discrete volume of fluids, reduces the sample size and is ideal for biological, chemical, and food processing [4, 5]. The separation of diseased cells (such as cancer, sickle cell anemia, malaria infection, etc.) from the normal ones can be achieved using microfluidic techniques [3]. Further, coalescence of droplets has widespread applications viz., micro reactor, mixing of reagents in chemical and pharmaceutical industries [6].

In general, microfluidics methods can be classified broadly in to two categories, viz., passive and active techniques [2]. In passive methods, the characteristics of fluid flow, fluid properties and the microchannel design are primarily utilized for manipulation of micron-sized objects suspended in fluids without any application of external forces. On the other hand, the suspensions in fluid is subjected to an external perturbation which in turn drives the flow. Although passive methods are cheaper as it doesn't require any external forcing, however, high throughput can be achieved using active methods.

Active methods for micro flow handling are based on the use of external forcing via electric, magnetic, optical and acoustics [7]. Despite the significant advancement in active microfluidics methods, the development of an efficient technique that is gentle, contact-less and biocompatible continues to remain a challenge. The ability to use acoustic waves to manipulate microparticles solely based on their mechanical properties in microfluidics is known as acousto-microfluidics which is proven to possess the above characteristics [8]. An acoustofluidics device is consists of different solid substrates, i.e., silicon substrates where the microchannel is fabricated, a glass surface to cover microchannel from the top, a transducer which is attached to the silicone surface for actuation. A typical acoustofluidics device setup is shown in Figure 1.

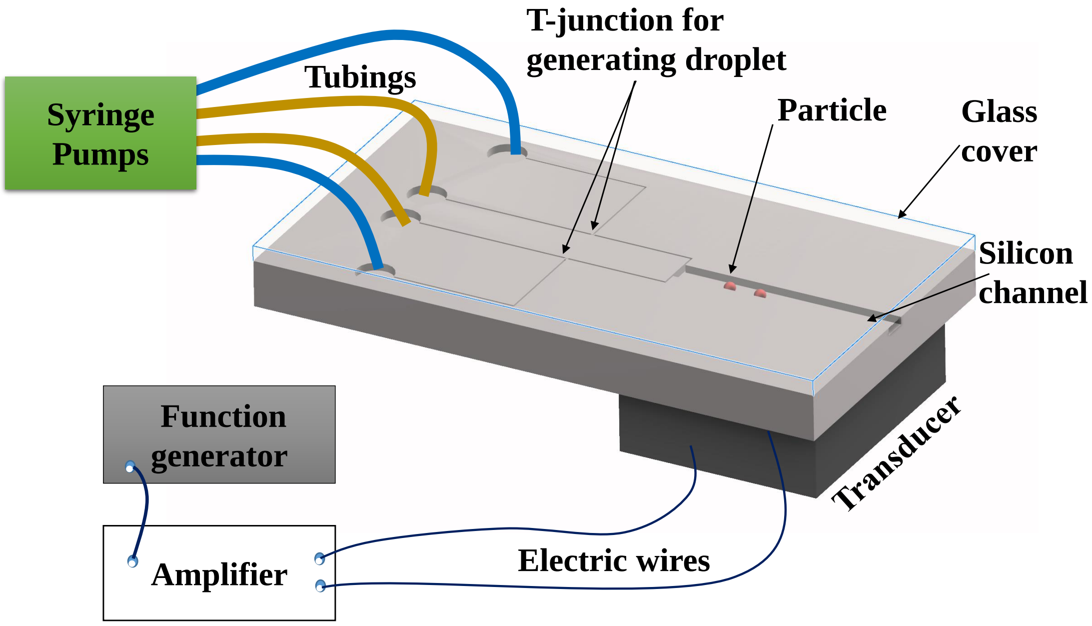

Figure 1. Schematic of an acousto-microfluidics device. Syringe pumps are used to flow fluids and suspensions inside the microchannel. The transducer is actuated using a function generator at the resonance frequency and the signal is amplified using an amplifier.

Usually, acousto-microfluidics devices are operated at half resonance modes, i.e., the channel width of is taken as half of the acoustic wavelength. Since, microfluidics devices are sizes of hundred of microns, ultrasound waves in the low MHz range is best suited for microfluidics applications [9]. For example, if the frequency of actuation is $f \geq 1.5$ MHz, considering water as the fluid (i.e., speed of sound $C_{wa} \approx 1.5 \times 10^3$ ms^{-1}), the wavelength obtained is about $\lambda_{wa} \leq 1$ mm. Therefore, the micon range half resonance mode can be achieved which may fit in to the submilimeter channels and cavities. Use of resonance modes are advantageous as they are usually stable and reproducible. We can control the spatial pattern formed by the resonance modes (pressure nodes and antinodes formation). Further, the maximum acoustic power defined in terms of acoustic energy density is delivered from the transducer at the resonance.

Here, we will be deriving the acoustic wave equation using regular first order perturbation theory. We will also discussed about the acoustic streaming field using the second order perturbation theory. Prediction of first and second order acoustic fields are shown in the next section. Finally, a brief summary and discussion is presented in Section 4.

2. First order perturbation theory: the wave equation

Sound waves require a medium for propagation whereas the light waves can travel in vacuum. The medium is generally considered as a fluid medium although sound can travel through solid object.

Therefore, acoustics is actually a special case of fluid dynamics. The acoustic wave equation should be derived from the fundamental equation of fluid dynamics. The basic equations of fluid mechanics are,

$$p = p(\rho), \tag{2.1}$$

$$\partial_t \rho = -\nabla.(\rho \mathbf{v}), \tag{2.2}$$

and

$$\rho \partial_t \mathbf{v} = -\nabla p - \rho(\mathbf{v}.\nabla)\mathbf{v} + \eta \nabla^2 \mathbf{v} + \beta\eta\nabla(\nabla.\mathbf{v}). \tag{2.3}$$

Before applying acoustic wave, let us consider a quiescent (steady) fluid with constant density ρ_0 and pressure p_0. Let an acoustic wave constitute tiny perturbation (subscript 1) in the fields of density ρ, pressure p and velocity $\mathbf{v}$.

$$\rho = \rho_0 + \rho_1, \qquad p = p_0 + p_1 = p_0 + c_0^2 \rho_1, \qquad \mathbf{v} = \mathbf{0} + \mathbf{v_1}. \tag{2.4}$$

Since sound wave is adiabatic, we perform the isentropic expansion of pressure about equilibrium,

$$p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_0 (\rho - \rho_0) + \cdots \qquad = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_0 \rho_1 + \cdots = p_0 + c_0^2 \rho_1 + \cdots. \tag{2.5}$$

Now from Equation 2.2,

$$\begin{aligned} \partial_t (\rho_0 + \rho_1) &= -\nabla. \left[(\rho_0 + \rho_1)\mathbf{v_1}\right] \\ &= \partial_t \rho_1 = -\rho_0 \nabla.\mathbf{v_1} - \nabla.(\rho_1 \mathbf{v_1}). \end{aligned}$$

Neglecting the higher order term (product of the first order term) we have,

$$\partial_t \rho_1 = -\rho_0 \nabla.\mathbf{v_1}. \tag{2.6}$$

From Equation (2.3),

$$\begin{aligned} (\rho_0 + \rho_1)\partial_t \mathbf{v_1} &= -\nabla\left(p_0 + c_0^2 \rho_1\right) - (\rho_0 + \rho_1)(\mathbf{v_1}.\nabla)\mathbf{v_1} + \eta\nabla^2 \mathbf{v_1} + \beta\eta\nabla(\nabla.\mathbf{v_1}) \\ => \rho_0 \partial_t \mathbf{v_1} + \rho_1 \partial_t \mathbf{v_1} &= -c_0^2 \nabla \rho_1 - \rho_0 (\mathbf{v_1}.\nabla)\mathbf{v_1} - \rho_1 (\mathbf{v_1}.\nabla)\mathbf{v_1} + \eta\nabla^2 \mathbf{v_1} + \beta\eta\nabla(\nabla.\mathbf{v_1}). \end{aligned}$$

Again, neglecting the product of first order term we get,

$$\rho_0 \partial_t \mathbf{v_1} = -c_0^2 \nabla \rho_1 + \eta \nabla^2 \mathbf{v_1} + \beta\eta\nabla(\nabla.\mathbf{v_1}). \tag{2.7}$$

Now, taking time derivative of Equation 2.6 and using Equation (2.7), we have

$$\begin{aligned} \partial_t^2 \rho_1 &= -\nabla.(\rho_0 \partial_t \mathbf{v_1}) \\ &= -\nabla.\left[-c_0^2 \nabla \rho_1 + \eta\nabla^2 \mathbf{v_1} + \beta\eta\nabla(\nabla.\mathbf{v_1})\right] \\ &= c_0^2 \nabla^2 \rho_1 - \eta\nabla^2(\nabla.\mathbf{v_1}) - \beta\eta\nabla^2(\nabla.\mathbf{v_1}) \\ &= c_0^2 \nabla^2 \rho_1 - \eta(1+\beta)\nabla^2(\nabla.\mathbf{v_1}) \\ &= c_0^2 \nabla^2 \rho_1 - \frac{\eta(1+\beta)}{-\rho_0}\nabla^2(\partial_t \rho_1) \\ &= c_0^2 \left(1 + \frac{\eta(1+\beta)}{\rho_0 c_0^2}\partial_t\right)\nabla^2 \rho_1. \end{aligned}$$

Therefore,

$$\partial_t^2 \rho_1 = c_0^2 \left[1 + \frac{(1+\beta)\eta}{\rho_0 c_0^2}\partial_t\right] \nabla^2 \rho_1. \tag{2.8}$$

Let us assume harmonic time dependence of all fields,

$$\rho_1 = \rho_1(\mathbf{r})e^{-i\omega t}, \qquad p_1 = c_0^2 \rho_1(\mathbf{r})e^{-i\omega t}, \qquad \mathrm{v}_1 = v_1(\mathbf{r})e^{-i\omega t}. \tag{2.9}$$

where $\omega = 2\pi f$ is the angular frequency and f is the frequency of the acoustic field. Using the harmonic fields equation we can further simplify the Equation (2.8),

$$\begin{aligned}
\partial_t^2 \left(\rho_1 e^{-i\omega t}\right) &= c_0^2 \left[1 + \frac{(1+\beta)\eta}{\rho_0 c_0^2}\partial_t\right] \nabla^2 \left(\rho_1 e^{-i\omega t}\right) \\
=> -\omega^2 \left(\rho_1 e^{-i\omega t}\right) &= c_0^2 \left[1 + \frac{(1+\beta)\eta}{\rho_0 c_0^2} \times (-i\omega) e^{-i\omega t}\right] \nabla^2 \left(\rho_1\right) \\
=> -\frac{\omega^2}{c_0^2} p_1 &= \left[1 - i\frac{(1+\beta)\eta\omega}{\rho_0 c_0^2}\right] \nabla^2 \left(c_0^2 \rho_1 e^{-i\omega t}\right) \\
=> -\frac{\omega^2}{c_0^2} p_1 &= \left[1 - i\frac{(1+\beta)\eta\omega}{\rho_0 c_0^2}\right] \nabla^2 p_1 \\
=> -k_0^2 p_1 &= \left[1 - i2\gamma\right] \nabla^2 p_1,
\end{aligned}$$

where k_0, a real valued wave number and γ, acoustic damping factor defined by

$$k_0 = \omega / c_0 \tag{2.10}$$

and

$$\gamma = \frac{(1+\beta)\eta\omega}{2\rho_0 c_0^2}. \tag{2.11}$$

For smallness of γ, we can approximate $[1 - i2\gamma] \approx [1 + i\gamma]^{-2}$ and we have,

$$\begin{aligned}
-k_0^2 p_1 &= [1 - i2\gamma]\, \nabla^2 p_1 \\
=> -k_0^2 p_1 &= [1 + i\gamma]^{-2} \nabla^2 p_1 \\
=> \nabla^2 p_1 &= -k_0^2 [1 + i\gamma]^2 p_1 \\
=> \nabla^2 p_1 &= -k^2 p_1,
\end{aligned}$$

where $k = k_0[1 + i\gamma]$ is the complex wave number. The Helmholtz equation for damped waves is given as,

$$\nabla^2 p_1 = -k^2 p_1. \tag{2.12}$$

As $\gamma \ll 1$, we can neglect the viscosity of the bulk part of the acoustic wave. Then we have,

$$\begin{aligned}
\nabla^2 p_1 &= -k_0^2 p_1 \\
=> \nabla^2 p_1 &= -\frac{\omega^2}{c_0^2} p_1.
\end{aligned}$$

Since, $\partial_t e^{-i\omega t} = -i\omega e^{-i\omega t} => \partial_t \sim -i\omega => i\partial_t \sim \omega$, we can modify the above equation as,

$$\begin{aligned}
\nabla^2 p_1 &= -\frac{(i\partial_t)(i\partial_t)}{c_0^2} p_1 \\
\nabla^2 p_1 &= \frac{1}{c_0^2}\partial_t^2 p_1.
\end{aligned}$$

For non-viscous flows, we have the more familiar wave equation as,

$$\nabla^2 p_1 = \frac{1}{c_0^2}\partial_t^2 p_1. \tag{2.13}$$

The solution of 1D wave equation is of form $p_1(x,t) = p_1(x \pm c_0 t)$. Pressure perturbation at $t = 0$ propagates a distance $\pm c_0 t$ in time t, c_0 indeed can be interpreted as the speed of sound.

In the inviscid limit and using the harmonic time dependence field, Equation (2.7) can be written as,

$$\begin{aligned} \rho_0 \partial_t \mathbf{v_1} &= -c_0^2 \nabla \rho_1 \\ => \rho_0(-i\omega)v_1 e^{-i\omega t} &= -c_0^2 \nabla \rho_1 e^{-i\omega t} \\ => -\rho_0 i\omega v_1 &= -\nabla(c_0^2 \rho_1) \\ => v_1 &= \frac{1}{i\rho_0\omega}\nabla(p_1) \\ => v_1 &= -\frac{i}{\rho_0\omega}\nabla(p_1). \end{aligned}$$

For a inviscid flow, we can defined velocity potential as $v = \nabla\phi$. Therefore,

$$\begin{aligned} v_1 &= -\frac{i}{\rho_0\omega}\nabla p_1 \\ => \nabla\phi_1 &= -\frac{i}{\rho_0\omega}\nabla p_1 \\ => \nabla\phi_1 &= -\nabla\left(\frac{i}{\rho_0\omega}p_1\right) \\ => \phi_1 &= -\frac{i}{\rho_0\omega}p_1. \end{aligned}$$

The relation between first order velocity potential and pressure is given by,

$$\phi_1 = -\frac{i}{\rho_0\omega}p_1. \tag{2.14}$$

Thus, both velocity and density can be calculated from the pressure field p_1.

2.1. First order equations

The first order continuity and momentum equations are given by Equation 2.6 and Equation 2.7. Usually, the acoustic fields are excited by a single-frequency vibration of the boundaries. When the system has stabilized in a steady oscillatory state, the first-order fields can be described by pure harmonics, oscillating with the excitation frequency ω. The solution can then be expressed in the frequency domain instead of the time domain, and we use the complex notation

$$g_1(\mathbf{r},t) = Re\left[g_1(\mathbf{r})e^{-i\omega t}\right], \tag{2.15}$$

and

$$\partial_t g_1 = -i\omega g_1. \tag{2.16}$$

Considering only the steady state solution, the first order equations can be transformed from time domain to frequency domain,

$$-i\omega\rho_1 = -\rho_0\nabla.\mathbf{v_1} \tag{2.17a}$$

$$-i\omega\rho_0\mathbf{v_1} = -c_0^2\nabla\rho_1 + \eta\nabla^2\mathbf{v_1} + \beta\eta\nabla\left(\nabla.\mathbf{v_1}\right). \tag{2.17b}$$

Equation (2.17) together with a set of boundary conditions constitute a steady-state first-order acoustic problem under the assumptions of adiabatic thermodynamics and single frequency vibrations of the boundaries.

2.2. The wave equation for the first-order velocity field $\mathbf{v_1}$

The wave equation for the first-order velocity field $\mathbf{v_1}$ is in general not simple to establish. However, in the special case of zero rotation *i.e.*, $\nabla \times \mathbf{v_1} = 0$. Let us consider the vector identity,

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{v_1}) &= \nabla(\nabla.\mathbf{v_1}) - \nabla^2\mathbf{v_1}\\ => 0 &= \nabla(\nabla.\mathbf{v_1}) - \nabla^2\mathbf{v_1}\\ => \nabla^2\mathbf{v_1} &= \nabla(\nabla.\mathbf{v_1}).\end{aligned}$$

Let us consider the first order continuity equation,

$$\partial_t\rho_1 = -\rho_0(\nabla.\mathbf{v_1}).$$

For harmonic time dependent first order density field, $\rho_1 = \rho_1 e^{-i\omega t}$ we have,

$$\begin{aligned}\partial_t\rho_1 &= \rho_1(-i\omega)e^{-i\omega t}\\ &= -\frac{i\omega}{c_0^2}(c_0^2\rho_1 e^{-i\omega t})\\ &= -\frac{i\omega}{c_0^2}p_1.\end{aligned}$$

Therefore,

$$\begin{aligned}-\rho_0(\nabla.\mathbf{v_1}) &= -\frac{i\omega}{c_0^2}p_1\\ => \nabla.\mathbf{v_1} &= \frac{i\omega}{\rho_0 c_0^2}p_1.\end{aligned}$$

Now, considering the Equation (2.7) and assuming harmonic time dependent velocity field we have,

$$\begin{aligned}
\rho_0 \partial_t \mathbf{v_1} &= -c_0^2 \nabla \rho_1 + \eta \nabla^2 \mathbf{v_1} + \beta \eta \nabla (\nabla . \mathbf{v_1}) \\
=> \rho_0 (-i\omega) e^{-i\omega t} v_1 &= -\nabla (c_0^2 \rho_1) + \eta \nabla (\nabla . \mathbf{v_1}) + \beta \eta \nabla (\nabla . \mathbf{v_1}) \\
=> \rho_0 (-i\omega) \mathbf{v_1} &= -\nabla (p_1) + (\eta + \beta \eta) \nabla (\nabla . \mathbf{v_1}) \\
=> \rho_0 (-i\omega) \mathbf{v_1} &= -\nabla (p_1) + (\eta + \beta \eta) \nabla \left(\frac{i\omega}{\rho_0 c_0^2} p_1 \right) \\
=> \rho_0 (-i\omega) \mathbf{v_1} &= -\nabla (p_1) + (i 2\gamma) \nabla p_1 \\
=> \rho_0 (-i\omega) \mathbf{v_1} &= -\left(1 - i2\gamma\right) \nabla p_1 \\
=> \mathbf{v_1} &= \frac{-i}{\rho_0 \omega \left(1 + i\gamma\right)^2} \nabla p_1 \\
=> \nabla \phi_1 &= \frac{-i}{\rho_0 \omega \left(1 + i\gamma\right)^2} \nabla p_1 \\
=> \phi_1 &= \frac{-i}{\rho_0 \omega \left(1 + i\gamma\right)^2} p_1 .
\end{aligned}$$

3. Second-order perturbation theory: acoustic streaming

There are two time scales involved in acousto-microfluidics problem. One is the fast time scales of the order μs which occurs due to the acoustic oscillation in the fluid medium and the other is the slowly evolving time scales due to the time averaged acoustic field. Since first order time-averaged field vanishes to zero, it is utmost important to understand the second order field using the perturbation analysis,

$$p = p_0 + p_1 + p_2 \qquad \rho = \rho_0 + \rho_1 + \rho_2 \qquad \mathbf{v} = \mathbf{0} + \mathbf{v_1} + \mathbf{v_2}. \tag{3.1}$$

Here, all the zeroth order and first order terms are assumed to be known. Neglecting the higher order term,

$$\begin{aligned}
p &= p_0 + \left(\frac{\partial p}{\partial \rho}\right)_0 (\rho - \rho_0) + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2}\right)_0 (\rho - \rho_0)^2 + \dots \\
&= p_0 + \left(\frac{\partial p}{\partial \rho}\right)_0 (\rho_1 + \rho_2) + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2}\right)_0 (\rho_1 + \rho_2)^2 + \dots \\
&= p_0 + \left(\frac{\partial p}{\partial \rho}\right)_0 \rho_1 + \left(\frac{\partial p}{\partial \rho}\right)_0 \rho_2 + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2}\right)_0 \left(\rho_1^2 + \rho_2^2 + 2\rho_1 \rho_2\right) + \dots \\
&= p_0 + \underbrace{\left(\frac{\partial p}{\partial \rho}\right)_0 \rho_1}_{p_1} + \underbrace{\left(\frac{\partial p}{\partial \rho}\right)_0 \rho_2 + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2}\right)_0 \rho_1^2}_{p_2} + \dots .
\end{aligned}$$

Therefore, the second order equation of state is given as,

$$p_2 = \left(\frac{\partial p}{\partial \rho}\right)_0 \rho_2 + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2}\right)_0 \rho_1^2 . \tag{3.2}$$

Here, we can define dimensional pre-factor $\frac{\rho_0 \left(\partial_\rho^2 p\right)_0}{\left(\partial_\rho p\right)_0}$ which is known as a non-linear parameter $\gamma^* - 1$ of the fluid,

$$\frac{\rho_0 \left(\partial_\rho^2 p\right)_0}{\left(\partial_\rho p\right)_0} = \gamma^* - 1. \tag{3.3}$$

Therefore, from Equation (3.2) it follows,

$$p_2 = c_0^2\rho_2 + \frac{1}{2}(\gamma^* - 1)\frac{c_0^2}{\rho_0}\rho_1^2. \tag{3.4}$$

Considering the first order continuity equation (Equation (2.6)), we proceed from Equation (2.2) as follows,

$$\begin{aligned}
\partial_t\rho &= -\nabla.(\rho\mathbf{v}) \\
=> \partial_t(\rho_0+\rho_1+\rho_2) &= -\nabla.[(\rho_0+\rho_1+\rho_2)(\mathbf{v_1}+\mathbf{v_2})] \\
=> \partial_t\rho_1 + \partial_t\rho_2 &= -\rho_0\nabla.\mathbf{v_1} - \rho_0\nabla.\mathbf{v_2} - \nabla.(\rho_1\mathbf{v_1}) - \nabla.(\rho_1\mathbf{v_2}) - \nabla.(\rho_2\mathbf{v_1}) - \nabla.(\rho_2\mathbf{v_2}) \\
=> \underbrace{\partial_t\rho_1 + \rho_0\nabla.\mathbf{v_1}}_{\text{Equation (2.6)}} + \partial_t\rho_2 &= -\rho_0\nabla.\mathbf{v_2} - \nabla.(\rho_1\mathbf{v_1}) - \underbrace{\nabla.(\rho_1\mathbf{v_2}) - \nabla.(\rho_2\mathbf{v_1}) - \nabla.(\rho_2\mathbf{v_2})}_{\text{higher order term}} \\
=> \partial_t\rho_2 &= -\rho_0\nabla.\mathbf{v_2} - \nabla.(\rho_1\mathbf{v_1}).
\end{aligned}$$

The second order continuity equation is given by,

$$\partial_t\rho_2 = -\rho_0\nabla.\mathbf{v_2} - \nabla.(\rho_1\mathbf{v_1}). \tag{3.5}$$

Similarly, using the Equation (2.7) and neglecting the higher order terms(collecting only the second order terms) we have the second order momentum equation as,

$$\rho_0\partial_t\mathbf{v_2} = -\nabla p_2 + \eta\nabla^2\mathbf{v_2} + \beta\eta\nabla(\nabla.\mathbf{v_2}) - \rho_1\partial_t\mathbf{v_1} - \rho_0(\mathbf{v_1}.\nabla)(\mathbf{v_1}). \tag{3.6}$$

Equations (3.5) and (3.6) can be further split into two sets of equation. In steady state the second-order variables consist of a steady component and an oscillatory component oscillating at 2ω, similar to product of two sines, $sin(\omega t)sin(\omega t) = \frac{1}{2} - \frac{1}{2}cos(2\omega t)$. The steady component is denoted by superscript "0" and the oscillatory second-order component is denoted by superscript "2ω",

$$g_2(\mathbf{r},t) = (g_2(\mathbf{r},t))^0 + (g_2(\mathbf{r},t))^{2\omega} \tag{3.7a}$$

$$= < g_2(\mathbf{r},t) > + Re\left(g_2^{2\omega}(\mathbf{r})e^{-i2\omega t}\right). \tag{3.7b}$$

This decomposition is valid only when considering a steady periodic state, and it is essentially a temporal Fourier decomposition of the second-order fields. $< g_2 >$ denotes time-averaging over one oscillation period $t_0 = \frac{2\pi}{\omega}$ and in steady state it equals the zero-order temporal Fourier component of the field

$$(g_2(\mathbf{r},t))^0 = < g_2(\mathbf{r},t) > = \frac{1}{t_0}\int_{t-t_0/2}^{t+t_0/2} g_2(\mathbf{r},t')dt' \tag{3.8}$$

$g_2^{2\omega}(\mathbf{r})$ is the complex amplitude of the secondary oscillatory mode, equivalent to $g_1(\mathbf{r})$ in Equation (2.15), and is given by the second-order Fourier component

$$g_2^{2\omega}(\mathbf{r}) = \frac{1}{t_0}\int_{t-t_0/2}^{t+t_0/2} g_2(\mathbf{r},t')e^{-i2\omega t'}dt'. \tag{3.9}$$

3.1. The time average of a product of time-dependent functions

Consider the real physical quantities $A(t)$ and $B(t)$ with harmonic time variation,

$$A(t) = Re[A_0 e^{-i\omega t}], \qquad B(t) = Re[B_0 e^{-i\omega t}], \tag{3.10}$$

where A_0 and B_0 are complex amplitudes. Rewriting $A(t)$ and $B(t)$ as follows,

$$A(t) = \frac{1}{2}\left[A_0 e^{-i\omega t} + A_0^* e^{i\omega t}\right], \qquad B(t) = \frac{1}{2}\left[B_0 e^{-i\omega t} + B_0^* e^{i\omega t}\right].$$

We find the time average,

$$\begin{aligned} < A(t)B(t) > &= \frac{1}{4\tau}\int_0^\tau dt \left[A_0 e^{-i\omega t} + A_0^* e^{i\omega t}\right]\left[B_0 e^{-i\omega t} + B_0^* e^{i\omega t}\right] \\ &= \frac{1}{4\tau}\int_0^\tau dt \left[A_0 B_0^* + A_0^* B_0 + A_0 B_0 e^{-i2\omega t} + A_0^* B_0^* e^{i2\omega t}\right] \\ &= \frac{1}{4}\left[A_0 B_0^* + A_0^* B_0\right] = \frac{1}{2}\mathrm{Re}\left[A_0 B_0^*\right]. \end{aligned}$$

Here A and B could be any first-order fields. This can be used to decompose the second order equations into one set of equations governing the steady component and one set of equations governing the oscillatory component of the second-order fields. The second order continuity equation is thus in steady state separated into

$$0 = -\rho_0 \nabla. < \mathbf{v}_2 > -\nabla. < \rho_1 \mathbf{v}_1 > \tag{3.11a}$$

$$-i2\omega\rho_2^{2w} = -\rho_0 \nabla.\mathbf{v}_2^{2\omega} - \nabla.(\rho_1 \mathbf{v}_1)^{2\omega} \tag{3.11b}$$

where we have utilized that $< \partial_t g_2 >= 0$ for any steady-state second order field, and $\partial_t g_2^{2\omega} = -i2\omega g_2^{2\omega}$. Similarly, the second-order momentum Equation 3.6 separates into

$$< \rho_1(-i\omega\mathbf{v}_1) > +\rho_0 < (\mathbf{v}_1.\nabla)\mathbf{v}_1 > = -\nabla < p_2 > +\eta\nabla^2 < \mathbf{v_2} > +\beta\eta\nabla(\nabla. < \mathbf{v_2} >) \tag{3.12a}$$

$$-i2\omega\rho_0\mathbf{v}_2^{2\omega} + (\rho_1(-i\omega\mathbf{v}_1))^{2\omega} + \rho_0((\mathbf{v}_1.\nabla)\mathbf{v}_1))^{2\omega} = -\nabla p_2^{2\omega} + \eta\nabla^2\mathbf{v}_2^{2\omega} + \beta\eta\nabla(\nabla.\mathbf{v}_2^{2\omega}). \tag{3.12b}$$

Equations (3.11b) and (3.12) together with a set of boundary conditions constitute a steady-state second-order acoustic problem, under the assumptions of adiabatic thermodynamics and single frequency vibrations of the boundaries. In the bulk fluid, where viscosity can be neglected Equation (3.12) can be written as,

$$< \rho_1(-i\omega\mathbf{v}_1) > +\rho_0 < (\mathbf{v}_1.\nabla)\mathbf{v}_1 > = -\nabla < p_2 > \tag{3.13a}$$

$$-i2\omega\rho_0\mathbf{v}_2^{2\omega} + (\rho_1(-i\omega\mathbf{v}_1))^{2\omega} + \rho_0((\mathbf{v}_1.\nabla)\mathbf{v}_1))^{2\omega} = -\nabla p_2^{2\omega}. \tag{3.13b}$$

As an example, let us consider a rectangular cross section of dimension 380 $\mu m \times 160$ μm. Using two-dimensional numerical simulation, the first order acoustic field and second order velocity field is calculated and is presented in Fig. 2. The ultrasound waves developed by actuating the transducer via signal generator and amplifier (see Figure 1). The width of the domain is kept is half wavelength so that the pressure node formation occur at the center of the microchannel (refer to Figure 2(a)). The second order acoustic velocity field created four symmetrical vortices inside the domain as seen in Figure 2(b).

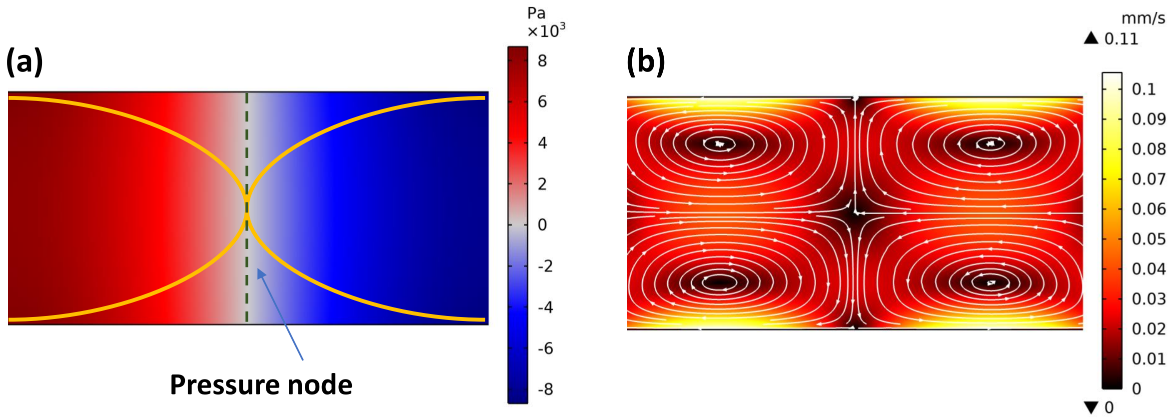

Figure 2. (a) first order acoustic pressure field, (b) Streaming velocity field at the 1.96 MHz resonating frequency in a rectangular cross section 380 $\mu m \times 160$ μm.

4. Conclusion

We shed light into the derivation of acoustic wave equation from the Navier-Stokes equation using first order perturbation theory. Further, we presented the acoustic streaming velocity field using second order velocity field. The acoustic wave theory is useful in understand the acoustic radiation forces that acts on the particle. The acoustic radiation forces can be utilized for manipulating microparticles or cells etc. inside a microchannel [10, 11].

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