Problem Section 7

Edited by Manjil P. Saikia

School of Mathematics, Cardiff University, CF24 4AG, UK

E-mail: `manjil@saikia.in`

This section contains unsolved problems, whose solutions we ask from the readers, which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *30 September, 2022*. If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

Problem 15. *Proposed by Manjil P. Saikia (Cardiff University)*

Prove without using the method of induction, that for any natural number n, we have the following inequality

$$n^{n/2} \leq n! \leq \frac{(n+1)^n}{2^n}.$$

Problem 16. *Proposed by Manjil P. Saikia (Cardiff University)*

For all even positive integers n with $n \geq 14$, show that

$$\sum_{k=1}^{\frac{n-6}{2}} \left\lfloor \frac{n-2k-2}{4} \right\rfloor > 1 + \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n-6k+2}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor.$$

For all odd positive integers n with $n \geq 11$, show that

$$\sum_{k=1}^{\frac{n-5}{2}} \left\lfloor \frac{n-2k-1}{4} \right\rfloor > 1 + \left\lfloor \frac{n-4}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n-6k-1}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n-6k-5}{4} \right\rfloor.$$

Solutions to Old Problems

We did not receive any solutions from the readers for Problem 12, and one solution for Problem 13. We give the solutions for these problems below. Problem 14 from Volume 72 is still open for solutions from the readers.

Solution 12. *Solved by the proposer. The solution below is by B. Sury (Indian Statistical Institute (Bengaluru).*

Write $P = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. If $\gcd(a,b,c) = 1$, we know there exists a 3×3 integer matrix P whose first row is (a,b,c). Write $P^{-1} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}$. Since $P = \mathrm{Adj}(P^{-1})$, we get

$$a = q_2 r_3 - q_3 r_2$$

$$b = p_2 r_3 - p_3 r_2$$

$$c = p_2 q_3 - p_3 q_2.$$

Therefore, if $B = \begin{pmatrix} p_2 & r_2 \\ -q_2 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} p_3 & r_3 \\ -q_3 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = BC - CB.$$

Finally, in general, if $(a,b,c) = d > 1$, then write $(a,b,c) = d(a', b', c')$. Putting

$$\begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} = B'C' - C'B',$$

we get

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = (dB')C' - C'(dB').$$

Solution 13. *Solved by Kuldeep Sarma. The solution below is by Kuldeep Sarma (Tezpur University).*

If $n = 1$, the solution is obvious. Otherwise it is enough to prove it for n a prime power. Hence $n = p^k$ where p is some prime and $k \geq 1$. If $k = 1$, then it follows from the fact that $F_p[x]$ is a domain, since F_p is a field. Now suppose that it is true for some $k \geq 1$ and assume that $fg \equiv 0\,(p^{k+1})$. Then $fg \equiv 0\,(p)$, hence without loss of generality, we may assume that $f = pF$ for some $F \in \mathbb{Z}[x]$. Also $pFg \equiv 0(p^{k+1}) \iff Fg \equiv 0(p^k)$. Then by mathematical induction, all products of coefficients of F, g are divisible by p^k. The result for f, g now follows, since $f = pF$.