The Tower of Hanoi and the Josephus Problem

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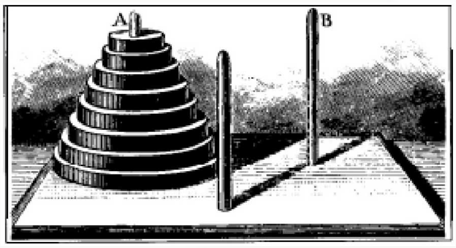
THE TOWER OF HANOI

The Tower of Hanoi is actually a well known puzzle in the field of mathematics. The puzzle tower of Hanoi was first invented by the French mathematician Edouard Lucas in 1883. Let us first look into the puzzle.



This puzzle consists of three tower pegs and more than one disks (here we will consider upto 8 disks). These disks are of different sizes and stacked upon in decreasing order i.e. the smaller one sits over the larger one. The main aim/mission of the puzzle is to move all the disks to some another tower without violating the sequence of arrangement. The below mentioned are few rules which are to be followed for completing the **Tower of Hanoi**:

(i) Only one disk can be moved among the towers at any given time.

(ii) Only the “top” disk can be removed.

(iii) No large disk can sit over a small disk.

At first this might seem to have a non trivial solution but as soon as we start taking out cases we will see a pattern. We will see the results for small numbers like $n = 1, 2, 3$ and then conclude the final result. Let us say that T_n is the minimum number of moves that will transfer n disks from one peg to another. Then it is obvious that $T_1 = 1$ & $T_2 = 3$. It is also clear that $T_0 = 0$.

Experiments with three disks show that the winning idea is to transfer the top two disks to the middle peg, then move the third, then bring the other two onto it. This gives us a clue for transferring n disks in general: we first transfer the $n-1$ smallest to a different peg (requiring T_{n-1} moves), then move the largest (requiring one move), and finally transfer the $n-1$ smallest back onto the largest (requiring another T_{n-1} moves). Thus we can transfer n disks (for $n > 0$) in at most $2T_{n-1}+1$ moves.

$$\mathbf{T_n \leq 2T_{n-1} + 1}, \text{ for } \mathbf{n > 0}.$$

This formula uses ' $\leq$ ' instead of ' $=$ ' because our construction proves only that $2T_{n-1}+1$ moves suffice; we haven't shown that $2T_{n-1}+1$ moves are necessary.

Similarly we will also look for the alternate case: At some point we must move the largest disk. When we do, the $n-1$ smallest must be on a single peg, and it has taken at least T_{n-1} moves to put them there. We might move the largest disk more than once, if we are not too alert. But after moving the largest disk for the last time, we must transfer the $n-1$ smallest disks (which must again be on a single peg) back onto the largest. This too requires T_{n-1} moves. Hence

$$\mathbf{T_n \geq 2T_{n-1} + 1}, \text{ for } \mathbf{n > 0}.$$

These two inequalities, together with the trivial solution for $n = 0$, yield

$$\mathbf{T_0 = 0; n = 0},$$

$$\mathbf{T_n = 2T_{n-1} + 1}, \text{ for } \mathbf{n > 0}.$$

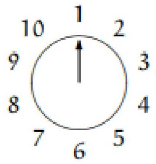
Now if we look for T_n, we will find a pattern for T_n, this pattern is generalized using the concept of recurrence relation. Recurrence relation gives a boundary value and an equation for the general value in terms of earlier ones. If we look at T_0, T_1, T_2, T_3 and so on we will find a relation i.e

$$\mathrm{T}_3 = 2.3 + 1 = 7; \mathrm{T}_4 = 2.7 + 1 = 15; \mathrm{T}_5 = 2.15 + 1 = 31; \mathrm{T}_6 = 2.31 + 1 = 63.$$

It certainly looks as if $\mathbf{T_n = 2^n - 1}$, for $n \geq 0$. This relation can very easily be proved by using the concept of Mathematical Induction.

THE JOSEPHUS PROBLEM

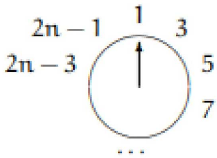
The Josephus problem is a real life application of mathematics. There is a very interesting story behind this problem, let us look into the story: Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted coconspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle. Let us generalize this concept and look into the problem: There are n people standing in a circle waiting to be executed. The counting out begins at some point in the circle and proceeds around the circle in a fixed direction. We start with n people numbered 1 to n around a circle, and we eliminate every second remaining person until only one survives. here's the starting configuration for $n = 10$.



The elimination order is $2, 4, 6, 8, 10, 3, 7, 1, 9$, so 5 survives.

The problem: Determine the survivor's number, i.e., J(n). Now as we saw for even number it seems like it might have some kind of pattern. We will try to generalize it and check it for odd numbers.

Let us consider 2n people originally. After a round the pattern will look like

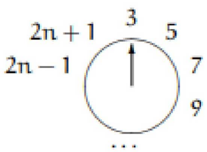


and 3 will be the next to go. This is just like starting out with n people, except that each person's number has been doubled and decreased by 1 . That is

$$\mathbf{J(2n) = 2J(n) - 1,\ \text{for}\ n \geq 1}$$

We can now go quickly to large n. For example, we know that J(10) = 5, so J(20) = 2 J(10) − 1 = $2 \cdot 5 - 1 = 9$.

Now we can look into the odd case, i.e $2n + 1$ people. With $2n + 1$ people, it turns out that person number 1 is wiped out just after person number $2n$, and we're left with



Again we almost have the original situation with n people, but this time their numbers are doubled and increased by 1. Thus

$$\mathbf{J(2n + 1) = 2J(n) + 1,\ \text{for}\ n \geq 1}$$

Combining these equations with J(1) = 1 gives us a recurrence that defines J in all cases:

$$\mathbf{J(1) = 1;\quad J(2n) = 2J(n) - 1,\ \text{for}\ n \geq 1; J(2n + 1) = 2J(n) + 1,\ \text{for}\ n \geq 1.}$$

For more information on these type of problems we refer the reader to the book [1].

References

[1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics.* Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.